"I tell you space is more plentiful than you think but it is far less substantial." — David Duncan, "Occam's Razor"

# THE RUBIK TESSERACT

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With the aid of a small computer belonging to one of us, we have worked out many properties of the four-dimensional counterpart of Rubik's Cube, which we call the Rubik Tesseract because it is based on the same geometrical principles that Rubik realized in his famous cube. It will be obvious that we have been guided by Dr. Singmaster's analysis of the Rubik cube and that our analysis could not have been made without studying his "Notes on Rubik's Magic Cube"(13).

## 1. Description of the Rubik Tesseract

A tesseract is a four–dimensional hypercube(7). It is bounded by eight cubical "faces" which meet at 16 corners, 32 edges, and 24 plane surfaces that we will call "squares", reserving the word "faces" for the boundary cubes. If we slice the tesseract into 81 equal smaller tesseracts, which we will call "tessies", and permit each of the eight "outer" layers of 27 tessies to rotate as a rigid body in a manner that preserves the shape of the main tesseract, then we have a Rubik tesseract. (The rotations will be defined more precisely in the next section.)

One tessie is at the center of the main tesseract, and one of them lies at the center of each of the eight rotatable layers. These nine tessies retain their positions in a rotation; the other 72 can be moved from one position to another. As with the Rubik cube, the re-arrangements can be very complex.

Each intersection of two or more faces of the main tesseract is the site of a movable tessie, and the (cubical) boundaries of that tessie which lie in the intersecting faces are its "facets". Since four faces meet at each corner of the tesseract, there are sixteen 4-faceted tessies, which we call "tetrads". Three faces meet at each edge, so there are 32 3-faceted tessies which we call "triads"; and two faces meet at each square, so there are 24 2-faceted tessies that we call "dyads". Altogether the 72 movable tessies have 208 facets.

When the tesseract is sliced into tessies, each of its cubical faces is divided into 27 smaller cubes like the "cubies" of a Rubik cube (if the cubies were real cubes instead of only having the external appearance of cubes). One of these cubes is at the center of the face and the other 26 are facets of some of the movable tessies. That is, if we think of a face as a Rubik cube, the eight corner cubies are facets of tetrads, the twelve edge cubies are facets of triads, and the six axial cubies are facets of dyads. Since each facet belongs to a different tessie, there are 26 movable tessies in the layer adjacent to a given face, and rotations of this layer will involve only these 26 tessies.

Many people find it difficult, perhaps impossible, to visualize a tesseract (although it is not different in principle from trying to visualize a three-dimensional machine from blueprints of it). But its faces can be visualized in three dimensions by rotating them into a common 3-space, in much the same manner that the surface of a cube can be developed onto a plane surface. Salvador Dali used this idea in his famous painting "Corpus Hypercubus" (See Ref. 7) and we show such a representation in Figure 1. To avoid confusion only the facets of four tetradic positions are shown. Facets belonging to the same position are numbered consecutively; e.g. $145, 146, 147, 148$. Further details in this Figure will be explained in the subsequent discussion.

We number the faces of the tesseract 1 to 8 so that opposite faces are congruent, mod 4. (5 is opposite to

1, etc. It is sometimes convenient to denote the face opposite to 1 by $1'$, etc.) Then the position of a tessie can be designated by the faces in which its facets lie; for example, the tetradic position whose facets we have numbered 145-148 in Figure 1 is designated 1234. A triad will have a 3-digit number; for example, 127 designates a triad with facets in faces 1, 2 and 7. It could also be designated $123'$. Similarly, 12 designates a dyad.

## 2. Rotations of the Rubik Tesseract

The tesseract has four principal axes that meet orthogonally at its center and are perpendicular to the faces they intersect. The directed axes passing through faces 1 to 4 are considered positive and are numbered 1 to 4 also. The negative axes are 5 to 8, 5 being the negative 1-axis, etc. We sometimes call these axes -1 to -4. In Figure 1 we show the positive axes in each face, as they are rotated along with the face into the "3-space" of the picture.

In 2-space, a rotation occurs around a center (a point), in 3-space around an axis (a line). In 4-space it occurs around a plane that we will call the "plane of rotation" (not to be confused with the planes, perpendicular to it, in which points move through arcs under the rotation). For the Rubik tesseract, the permitted rotations are 90 degree rotations (or multiples thereof) of a layer of tessies around any of the planes of rotation formed by a pair of principal axes. The principal axis that "points to" the layer being turned is called the "primary axis"; it may be any one of the eight axes, positive or negative. The other principal axis that defines the plane of a rotation is called the "secondary axis" of the rotation.

The face of the tesseract to which the primary axis of a rotation points (and which bears the same number) turns as a whole, in its own 3-space, around an axis through its center parallel to the secondary axis of rotation. (Similarly, a rotation of a Rubik cube causes one of its faces to rotate in its own plane, about its center point.) If the rotation of this face is clockwise, as seen by an observer in the same 3–space who looks backward along the positive (translated) secondary axis from outside the cube, we call the Rubik rotation positive. A counterclockwise rotation is called negative. There are, therefore, three different positive Rubik rotations for each face of the tesseract, corresponding to the three axes about which the face can be rotated in its own 3–space.

A positive 90 degree rotation about a primary axis, $i$, and a secondary axis, $j$, will be called a basic rotation of the Rubik tesseract and will be denoted by $R_{ij}$. The inverse, or negative rotation, will be denoted by $R'_{ij}$. Now $i$ can have any value from 1 to 8 and $j$ any value from 1 to 4, but $j$ and $j+4$ must be different from $i$. So there are 24 basic rotations of the Rubik tesseract.

## 3. Rotations Expressed as Permutations of Tessies

Each basic rotation changes the positions of tessies having facets lying in the face of the primary axis, and the rotation can be described as a product of cyclic permutations of the tessie positions. It is evident from the manner in which the face rotates about the translated secondary axis that each basic rotation will involve two 4–cycles of tetrads, three 4–cycles of triads, and one 4–cycle of dyads, all disjoint. The other two dyads with facets in the face are in the plane of rotation (their facets in the face are on the translated secondary axis) and so they turn but do not move to new positions under the rotation. Altogether, 24 tessies and 76 facets change position under each rotation.

In Figure 1 we show the motion of the tetrad whose "Start" position (indicated by the colors of the faces) is 1234, under the action of the rotation $R_{12}$. The successive positions of the tetrad can be described by a single tessie cycle or by four cycles of facets, as shown on the Figure.

To obtain the 24 basic permutations we wrote a computer program which calculated the permutations of the 208 facets and converted them into permutations of tessies. Each facet was assigned a number from 1 to 208 and its location identified by a coordinate vector. The $i$-th coordinates of the facets lying in Face $i$ ($i = 1$ to 4) were assigned the value of 2, and -2 for the facets in Face $i+4$, or $-i$. The other coordinates of the facets in the same face were given values of 1, $-1$, or zero according to their locations in the tri–axial coordinate frame of the face. Thus, dyadic facets have two zero coordinates, triadic facets have one, and tetradic facets have none. For instance, Facet No. 160 in Figure 1 will be seen to have the coordinate vector $(1, 1, -1, -2)$.

Each rotation was described by a 4x4 rotation matrix(12). For instance, the rotation $R_{12}$ has the matrix

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{matrix}$$

for facets whose 1–component is positive, and the unit matrix for all other facets. Then the new location and facet number after a rotation was found by pre–multiplying the old coordinate vector by the rotation matrix. Thus, multiplying $(1, 1, -1, -2)$ by $R_{12}$ gives the vector $(1, 1, -2, 1)$, which Figure 1 shows to be the location of Facet 155.

The 24 basic rotations expressed as cyclic permutations are listed in Table 1. The digits in the first position of each cycle can be written in any order, but the subsequent positions must follow in corresponding order to maintain the proper orientations of the tessies.

Our nomenclature is similar to Singmaster's. We could, for example write

| | |
|---|---|
| 1 = Up | 5 = Down |
| 2 = Right | 6 = Left |
| 3 = Front | 7 = Back |
| 4 = In | 8 = Out |

However, we find that the numerical system is preferable for computer operation and for visualizing moves, and especially for turning the tesseract around in 4–space. (See Section 6a).

**4. Useful Moves - The Computer Model**

The foregoing discussion has been mainly a matter of definitions, preparatory to our main purpose, which is to analyze the moves on the Rubik tesseract in the manner that Singmaster analyzed the moves on the Rubik cube; that is to say, to determine what permutations of the tessies are possible, and to find "useful moves" that would permit any possible permutation to be accomplished. A "move" is any sequence of the basic rotations; a useful move is a move that involves only a small number of tessies, from which more complex permutations can be built up.

Since we cannot build a physical Rubik tesseract, we needed two tools to look for useful moves. The first was a device for visualizing the tesseract and suggesting useful moves; the second was a computer model to calculate the permutations resulting from the moves. Our visualizing device is simple: the diagram of Figure 2 mounted on a pin–board. This diagram is the "development" of the faces shown in Figure 1 with the faces moved apart so that they don't overlap. The locations of the facets of any tessie are easily found from the coordinate axes and marked with pins. A basic rotation $R_{ij}$ is followed by looking at face i and visualizing it turning clockwise around axis j. Thus the new position of any facets in that face can be found and marked, and the other facets of the same tessie in other faces can be found because they have the same set of face numbers. The pin–board diagram is especially useful for finding conjugating moves.

The computer model of the tesseract permits us to carry out moves and see the results. It contains an array (M) that describes the current arrangement of the tesseract. The model will perform the following routines on command:

> Re–arrange the M–array in accordance with an input move, that consists of basic rotations and stored combinations of basic rotations. The M–array can also be reset to the "Start" arrangement.
>
> Calculate and display the tessie cycles that produce the current arrangement from the "Start" arrangement

A program listing is included in Appendix B. The programming language is "Tiny Pascal"(5), which we selected for its suitability to the problem and its availability to us. For the final phases of this work, we purchased a version of Tiny Pascal which operates under CP/M (the most widely used operating system for 8–bit microcomputers). The conversion of the program to standard Pascal is relatively straightforward.

**a. Description of the Tesseract**

The model contains the 24 basic rotations in a data bank, in the form of the facet permutations that were calculated by the program described in Section 3. In the data bank, dyadic facets are numbered from 1 to 48, and facets of the same dyad are numbered consecutively; that is, facets 1 and 2 are on the same dyad, 3 and 4 are on the same dyad, etc. The triadic facets are numbered from 49 to 144, with the same system of consecutive numbering. The tetradic facets are numbered from 145 to 208, again with consecutive numbering. These values are stored in the array M, which is a vector of 208 elements. Initially, M[J] contains the value J, which means that each facet, and hence each tessie, is at the initial, or "Start" position.

After a move has been carried out, M[J] will contain the original location of the facet at location J. For example, a tesseract in the initial configuration operated on by the permutation (1, 5, 7, 11) results in M[1]=11, M[5]=1, M[7]=5, and M[11]=7. The M vector, then, is a complete description of the state of the tesseract.

**b. Moves**

Each basic rotation moves 24 tessies with their 76 facets to new locations in the tesseract. These 76 facets are permuted in 19 four-cycles. This permutation is carried out by PROC Z, which accesses the data bank and carries out each 4-cycle, using PROC PERMUTE to modify the M-vector. All of the basic rotations are in the CASE statement at the beginning of PROC Z. For a basic rotation $R_{ij}$, the argument N of PROC Z is the number $ij$. For rotations involving "negative" faces, two numbers are accepted: $-ij$, or $(i+4)j$. For example, Z(12) denotes a basic rotation of Face 1, with secondary axis 2. Z(-12) or Z(52) are equally acceptable means of indicating a basic rotation of the opposite face.

As a convenience, useful moves were added to PROC Z as they were found. The moves $Q_1$ to $Q_{36}$ (see following Sections and Appendix A) are called by using Z(101) to Z(136). Other moves are included with arguments of $200-425$; these are not intended to be used directly, but are required to carry out the $Q_i$ moves. All of these moves call on basic rotations, or call on other moves that ultimately use basic rotations. The recursion capability of Tiny Pascal is very useful in programming these operations.

**c. Program Control**

At any point, the effect of the moves that have been performed on the tesseract can be determined by entering the value 99. This causes the REKAP variable to be set TRUE, which causes the procedure GETCYCLES to execute, first for the dyads, then for the triads, and finally for the tetrads. The cycles are printed for each type of tessie. GETCYCLES is described below.

After the cycles are obtained, more moves may be made, and the cycles can be printed out for the tesseract in its new state. Or, the tesseract may be brought back to the initial state by entering 999, and a new set of operations begun.

**d. PROC GETCYCLES**

This procedure takes advantage of the fact that the facets of any single tessie are numbered consecutively. It begins by determining if any facet of a given tessie has been moved from the "Start" position. The variable ACTIVITY is set TRUE if this is the case. Then the permutations are traced out by using the procedure LOOKFOR, which recursively "chases" the original occupant of M[J] until it finds it, retaining along the way the locations it has traversed. When all the permutations of facets of a given tessie have been traced, the cycles are printed out. The facet numbers are not printed; instead the face numbers of the tesseract are used, so that the tessie notation described in Section 1 is the result. When orientation changes occur, it often happens that cycles of one facet differ in length from another facet of the same tessie. This is dealt with by revising the cycle length to a length equal to the least common multiple of all the cycles. Each cycle is then simply repeated the required number of times.

GETCYCLES starts with the lowest numbered facet, and works up. When a facet J has been involved in a permutation cycle, the variable FLAG[J] is set FALSE, and this facet is considered no further. This avoids repetition of permutation cycles for each tessie involved in the cycle.

The variable OK is used to indicate whether a legal move has been attempted. If a move is attempted which is not in the repertoire, an appropriate message is printed, and the user is prompted for another input move.

**e. Performance**

We use a computer with a Z80 processor, with a 4 MHz clock speed, and 56K bytes of programmable memory. The program in its present state carries out approximately 45 basic rotations per second. The longest move in PROC Z is $Q_{22}$, which consists of 14616 basic rotations - 323 seconds is required for this move. The program is operated in a 48K CP/M system, with the data bank occupying 2032 bytes above 48K (C000 hex). We have not attempted to operate with smaller storage than this. The Tiny Pascal system was purchased from Supersoft, Inc. They describe the compiler as requiring a minimum 36K CP/M system.

## 5. Permutations of Position

Even with the computer model and the pin-board, making moves on the tesseract is more cumbersome than making moves on a Rubik cube. But this disadvantage seems to be offset by the greater flexibility for making moves in four dimensions. At any rate, we were able to find all the moves necessary to generate any possible permutation, as detailed in Appendix A.

We first consider permutations of position. (Permutations of orientation are discussed in the next section.) From the cycle structure of the basic rotations, all such permutations of tetrads must be even. The positional permutations of triads and dyads considered by themselves can be even or odd, but taken together they must be even. The simplest possible permutation of tetrads, triads, or dyads by themselves is, therefore, a pair of interchanges.

By using the same idea that Singmaster suggested for the cube we were able to find very quickly a move, $Q_1$, that involved only two interchanges of dyads and two of triads. (It is rather remarkable that from this one move we were able to generate systematically all the moves, both positional permutations and orientation permutations, that are necessary to define the group of the Rubik tesseract.) From $Q_1$ it was fairly simple to find two interchanges of triads alone:

$$Q_4 = (238, 274)(638, 674)$$

and two interchanges of dyads alone:

$$Q_6 = (23, 27)(43, 47)$$

Finding two interchanges of tetrads was more involved but fairly straightforward. We arrived at:

$$Q_{17}^3 = (1346, 1836)(5678, 7654)$$

(The details of these moves are given in Appendix A.)

A move involving one dyadic and one triadic interchange can also be developed. (See Appendix A). In fact every possible permutation of positions can be obtained from the above three moves.

Although these moves are “simple” in terms of the permutations involved, they are not short. The last result above, for example, requires 1812 basic rotations. Nevertheless it is built up from $Q_1$ in only about ten short steps, and since the computer does the calculations in reasonable times, there is no great incentive to keep the moves short in terms of the number of basic rotations.

## 6. Permutations of Orientation

Permutations of orientation on the tesseract are even more interesting than those on the Rubik cube. They also come in a greater variety of types, so that Singmaster’s terminology is not adequate to describe them.

We have adopted a system in which a twisted cycle is indicated by letters $a, b, c, d$. For instance, $abc$ means that the first 3 facets of each tessie permute cyclically, the first to the position of the second, etc. Thus:

$$(123, 567)_{abc} = (123, 567, 312, 756, 231, 675)$$

$$(123, 567)_{ab} = (123, 567, 213, 657)$$

etc. We will call a permutation involving two letters a "flip" or a "reflection", one involving three letters a "twist", and one involving four letters, such as (ab)(cd), a "cross".

**a. Coordinate Systems**

Before discussing these permutations it is necessary to look more closely at the coordinate system for the tesseract. It is evident that we could have numbered the coordinate axes in numerous ways ($4! \times 2^4$ to be exact). These fall into two sets such that any two coordinate frames in the same set are superposable by turning the frame around in 4–space; but each frame in one set is an enantiomorph of one in the other set. [For more discussion of enantiomorphism see Ref. 6] In order to study changes in orientation we need to have a rule to know when two frames are congruous (i.e., superposable by rotations in 4-space). The rule is that the number of transpositions of positive axes plus the number of reversals of axes must be even. For instance, referring back to Section 3, the following frames are all congruous:

| | Up | Right | Front | In |
|---|---|---|---|---|
| Present Frame : | 1 | 2 | 3 | 4 |
| Exchange U–R and F–I : | 2 | 1 | 4 | 3 |
| Exchange F–I and Reverse F–B : | 1 | 2 | 8 | 3 |
| Reverse F–B and I–O : | 1 | 2 | 7 | 8 |

Thus, for example, it is possible to move a tetrad from 1234 to 1283 ($R_{12}$ does it) but impossible to move from 1234 to 1238. The same rule applies in three dimensions. (This is why a corner cubie can be twisted but not reflected.) In three dimensions it is not necessary to think about the rule because, when dealing with a physical object, it is impossible to violate it; but for four dimensions we must be aware of it. (Incidentally, an examination of the basic rotations in Table 1 shows that each tetradic move transposes one pair of axes and reverses one axis, in accordance with this rule. That is, each cycle is written congruously, as it must be. The two tetradic cycles of a rotation happen to be written incongruously to each other, but this does not matter.).

An application of this rule is the operation that we call "transpose" in Appendix A, which we employ to derive useful moves. The scheme is based on the fact that the digits in the tessie positions of a permutation are related to the subscripts on the rotations that generate the permutation. Thus we can get a new permutation merely by changing the subscripts of the rotation. For instance, the permutation (13, 14). . . can be changed to (23, 24). . . by interchanging 1 and 2 on the subscripts of the rotations involved. But, according to the above rule, if we make only one transposition we are looking at a "mirror image" of the tesseract, in which all rotations are reversed in direction. Thus, when we transpose two digits in the permutation, we must also invert all the rotations involved in the move (whether or not their subscripts are affected by the transposition).

We now consider the orientation-permutations for dyads, triads, and tetrads.

**b. Dyadic Flips**

Dyads have only two orientations. A reversal of orientation is called a flip, and since this is an odd permutation, it follows that flips must occur in pairs. In Appendix A we derive a pair of flips:

$$Q_{24} = (12)_{ab}(34)_{ab} = (12, 21)(34, 43)$$

Thus the situation for dyads is much the same as on the Rubik cube.

**c. Triadic Twists and Reflections - Gene Splicing**

Triads have six permutations of orientation corresponding to the $S_3$ group (symmetric group of permutations on three letters). Three of them are twists (abc, acb, and I=identity) and three are reflections (ab, ac, and bc). On the Rubik cube, only twists of corner cubies are possible because the reflections are 3-dimensional enantiomorphs. But just as two-dimensional enantiomorphs can be superposed in three dimensions, so 3-dimensional enantiomorphs, which the triadic reflections are, can be superposed in four dimensions. A reflection is an odd permutation, so they must occur in pairs. A twist is an even permutation, so they can occur in isolation. Such an isolated twist is derived, in Appendix A, by use of the transposition device described in Section 6a:

$$Q_{29} = (123)_{abc} = (123, 312, 231)$$

The reverse twist is $Q'_{29} = Q^2_{29} = (123)_{acb}$

We did not find any direct way to generate a pair of reflections, so we used a device we call "gene splicing" based on a rather far-fetched analogy with the recently developed techniques of genetic transformations. First, we found a long complex permutation (the "chromosome") that contains the desired reflection (the "gene"). We splice onto this chromosome another "gene", one of the isolated twists found above. Then we make a permutation that "mutates" the reflection into the twist and vice-versa, on the chromosome. Finally, by reacting this chromosome with the original one, we isolate the two genes, obtaining:

$$Q_{35} = (134)_{ac}(275)_{bc}$$

The details are in Appendix A.

**d. Tetradic Twists and Crosses**

There are 24 permutations of the four facets of a tetrad, comprising the $S_4$ group. Four of them are crosses [$(ab)(cd)$, $(ac)(bd)$, $(ad)(bc)$, and $I$], eight of them are twists, six are 2-cycles (e.g. $ab$, etc.), and six are 4-cycles (e.g. $abcd$, etc.). Two-cycles and 4-cycles cannot occur in 4-space because they violate the rule given above in Section 6a. That is, they involve an odd number of transpositions of axes and no reversals, so they are 4-dimensional enantiomorphs. For instance

$$(1234)_{cd} = (1234, 1243)$$

and this permutation cannot occur. Likewise, a 4–cycle is equivalent to an odd number of transpositions and can't occur either.

The even permutations, which form the alternating group ($A_4$), are all possible. [See Ref. 3 for explanation of the group-theoretic terminology used in the following analysis.] The crosses are a normal subgroup of the alternating group, that we shall call $N$. That is:

$$N = [I, (ab)(cd), (ac)(bd), (ad)(bc)]$$

The cosets of $N$, which we call $S$ and $Z$, are composed of the twists. Specifically

$$S = [abc, adb, acd, bdc]$$

$$Z = [acb, abd, adc, bcd]$$

The sets $N$ ,$S$, and $Z$ form the quotient-group of $A_4$ by $N$:

$$A_4/N = [N, S, Z]$$

in which $N$ acts as the identity element. The group multiplication table is:

$$NS = SN = Z^2 = S \qquad NZ = ZN = S^2 = Z \qquad SZ = ZS = N^2 = N$$

meaning that the product of an element from $N$ and one from $S$ is in $S$, the product of two elements of $Z$ is in $S$; the product of an element of $N$ and one from $Z$, or of two elements from $S$, is in $Z$; and the product of an element from $S$ and one from $Z$, or of two elements from $N$, is in $N$.

The quotient-group is cyclic and isomorphic with the group of residue classes, mod 3. If we count each $S$-twist as $+1$, each $Z$-twist as $-1$, and each cross as zero, then the above table states that in any interactions among the tetradic orientations the sum of all three types is invariant, mod 3. Since initially the sum is zero, the sum is always congruent to zero, mod 3. More simply put, the sum of $S$-twists ($+1$) and $Z$-twists ($-1$) must be congruent to zero, mod 3, but there is no constraint on the crosses. This is the equivalent, for the tetrads, of the rule for corner-cubies on the Rubik cube: that the sum of the clockwise ($+1$) and counterclockwise ($-1$) twists is congruent to zero, mod 3. But whereas the Rubik cube has only one twist of each type, the tesseract has four, not to mention the crosses; and they can be combined in any way subject to this constraint.

Therefore, an isolated cross is possible, but the minimum number of twists is two, of which one must be an $S$-twist and the other a $Z$-twist. Examples of both cases were found (see Appendix A). Thus:

$$Q_{21} = (1234)_{(ad)(bc)} = (1234, 4321)$$

$$Q_{18} = (1234)_{acd}(1283)_{adc}$$

In $Q_{18}$ we have another example of the rule we stated in Section 6a. For, $(1283)_{adc} = (1238)_{acd}$. If we wrote $Q_{18}$ in the form $(1234)_{acd}(1238)_{acd}$ we would seem to violate the $S - Z$ addition rule. But this is only because 1238 is on the "opposite side of the mirror" from 1234. It is exactly as if we twisted two corners of a Rubik cube and then looked at one of them directly and the other in a mirror; we would see two twists in the same direction. So, in expressions like $Q_{18}$, we ought to write the tessies congruously.

There are 32 possible pairs of twists on 1234 and 1283, with one being an $S$-twist and the other a $Z$-twist, and all of them can be obtained as shown in Appendix A. Similarly, the other two crosses on 1234 can be obtained.

**7. The Group of the Rubik Tesseract**

The group of the Rubik tesseract is composed of all those permutations that can be generated by the basic rotations in Table 1. We have shown that the constraints on this group are:

- Tetradic positional permutations must be even;
- Dyadic plus triadic positional permutations must be even;
- Dyadic flips and triadic reflections must be even;
- Tetradic $S$ - $Z$ twists must be congruent to zero, mod 3

We have also shown that all possible permutations within these constraints can actually be generated.

The number of permutations of position is therefore $(24!\,32!/2) \times (16!/2)$. The number of dyadic flips is $2^{24}/2$. With respect to the triads, any of the six orientations is possible for the first 31, but if the number of reflections to that point is even, the last orientation must be one of the three twists; and if it is odd, the last orientation must be one of the three reflections. So the total number of possibilities is $6^{32}/2$. Similarly, the first fifteen tetrads can have any of 12 orientations independently, but if the $S - Z$ sum at that point is zero, the last orientation must be one of the four crosses (including $I$), or if it is +1 the last orientation must be a $Z$-twist, or if it is -1, an $S$-twist. In each case, there are four possibilities for the last tetrad instead of 12, so the number of combinations is $12^{16}/3$.

So the total number of permutations, the order of the Rubik tesseract group, is $(24!\,32!\,16!\,/4) \times (2^{24}/2) \times (6^{32}/2) \times (12^{16}/3)$ or approximately $1.76 \times 10^{120}$, which is about the same as the number of ways to play a game of chess(2).

We have dealt only with very simple permutations and short cycles (except for the "chromosomes") but very long cycles and complex permutations are obviously possible. There are elements of order 12,432,420 ($= 2^2 \times 3^3 \times 5 \times 7 \times 11 \times 13 \times 23$), consisting of an 11-cycle and a 13-cycle of dyads, a 23-cycle and a twisted 9-cycle of triads, and a 5-cycle, a 7-cycle, and two crossed 2-cycles of tetrads. From cursory examination of the possibilities, this is probably the maximum order of an element in the group, but it might not be.

**8. Rubik N-topes**

"In that blessed region of Four Dimensions, shall we linger on the threshold of the Fifth, and not enter therein?" E. A. Abbott - "Flatland"

The foregoing ideas can obviously be extended to spaces of higher dimensions. For brevity, we shall call an $N$-dimensional regular orthogonal polytope an $N$-tope. The Rubik $N$-tope is obtained by slicing an $N$-tope into $3^N$ equal smaller $N$-topes, called "topies", and permitting any outer layer of $3^{N-1}$ topies to rotate rigidly in a manner analogous to the rotations of the Rubik cube and tesseract. The structural features of the Rubik $N$-tope are listed in Table 2.

The "axes" of a rotation of a Rubik $N$-tope are (N-2)-dimensional Euclidean spaces. For the 5-tope, for example, they are 3-spaces and there are six of them for each face, so that the Rubik 5-tope has sixty basic rotations.

The permutations of topies generated by the basic rotations form the group of the Rubik $N$-tope. The order of the group is calculated as follows. The basic rotations are always odd permutations of dyads and triads and are even permutations for $n$-ads of $n > 3$. (See at bottom of Table 2). So, the number of permutations of position is

$$P_p = \frac{(a_{N,2})!\,(a_{N,3})!\cdots(a_{N,N})!}{2^{N-2}}$$

where $a_{N,n}$ is the number of $n$-ads (see Table 2).

For $n < N$, the $n$-ads can have any orientations of the $S_n$ group (symmetric group on $n$ letters), provided that the total number of permutations of orientation is even; hence there are $(1/2)(n!\,)^{a_{N,n}}$ permutations of orientation of $n$-ads for $n < N$.

For the $N$-ads, the orientations are limited to the even permutations, that is, to the alternating group $A_N$, because odd permutions are enantiomorphic. The number of possible permutations depends on the number $t_N$ of orbits in the action of the $N$-tope group on the set of patterns of orientations of $N$-ads.

There are $(N!\,/2)^{a_{N,N}}$ such patterns. So the total number of permutations of orientation is:

$$P_o = \frac{(2!\,)^{a_{N,2}}(3!\,)^{a_{N,3}}\cdots(N!\,)^{a_{N,N}}}{t_N 2^{N-2+a_{N,N}}}$$

and the order of the Rubik N-tope group is $P_pP_o$. For $N$ equal to 3 or 4, the number of orbits, $t_N$, is equal to 3. We have seen in Section 6d that the reason there are three orbits for the case $N = 4$ is that the alternating group on four letters has a normal subgroup. For $N > 4$, $A_N$ has no normal subgroup; Buhler et al give a proof that $t_N = 1$ for $N > 4$ (Ref. 4).

For example, the order of the Rubik 5-tope group is

$$40!\,80!\,80!\,32!\,2^{34}6^{80}248060^{32} \approx 7.017 \times 10^{560}$$

It should be noted that this calculation gives only the maximum possible order for the Rubik $N$-tope group, since we have not actually demonstrated, as we did for $N = 4$, that all the permutations can actually be reached. But on the basis of the results for the tesseract we feel strongly that this is the actual order.

Like Mr. Square of Flatland(1), after this brief glimpse we must now descend from these insubstantial spaces.

### 9. The Rubik Cube Group as a Subgroup of the Rubik Tesseract Group

The Rubik cube group is a tiny subgroup of the Rubik tesseract group, considerably smaller, in terms of the relative numbers of their elements, than a proton is to the entire universe we know. Nevertheless, because of recent interest in the Rubik cube among a small part of humanity, we should take a look at this subgroup. After all, the structure of a proton is not without interest, at least to an even smaller part of humanity.

We have already noted that each face of a Rubik tesseract corresponds to a Rubik cube in the sense that each facet of the face is a cubie: tetradic facets are corner cubies, triadic facets are edge cubies, and dyadic facets are axial cubies. Now consider a particular face, say Face 4, and the rotations $R_{14}, R_{24}, R_{34}, R_{54}, R_{64}, R_{74}$ of the six faces adjoining it. The permutations generated by these rotations form a group in which no facets enter or leave Face 4, and their effects within Face 4 are exactly the same as the six basic rotations of the Rubik cube, except that they are left-handed. In fact each of the above rotations has one triadic and one tetradic cycle involving Face 4, and if we drop the 4 (which is is invariant) from each position of these cycles and neglect the other cycles not involving Face 4, and also drop the subscript 4 from the R's, we have

$$R_1 = (12, 17, 16, 13)(123, 172, 167, 136)$$

$$R_2 = (12, 32, 52, 72)(123, 325, 527, 721)$$

$$R_3 = (13, 63, 53, 23)(123, 613, 563, 253)$$

$$R_5 = (52, 53, 56, 57)(523, 536, 567, 572)$$

$$R_6 = (16, 76, 56, 36)(163, 761, 567, 365)$$

$$R_7 = (17, 27, 57, 67)(127, 257, 567, 617)$$

This is a set of basic (left-handed) rotations for the Rubik cube group. Additionally we note that the rotations $R_{41}, R_{42},$ and $R_{43}$ cause Face 4 to rotate about the secondary axis, in its own 3-space; that is, they turn the Rubik cube around. The rotation $R_{41}$, for example, turns the cube 90 degrees about Axis 1. The effect on the other axes is given by the dyadic cycle of $R_{41} : (24, 34, 64, 74)$. So Axis 2 goes to Axis 3, Axis 3 to Axis 6, etc. and we see that these are also left-handed turns.

### 10. Quarks

Viewing the Rubik cube as just a face of a Rubik tesseract throws a new light on the problem of "quark isolation" discussed by D. R. Hofstadter (10). In his article he describes an analogy (attributed to S. W. Golomb — see also Ref. 9) between corner–cubie twists and quarks. Like quarks, a twist cannot occur in isolation on a Rubik cube, but only in pairs of opposite twist or in three's with the same twist.

From our present vantage point, we see that the corner–cubie twists are mere shadows of tetradic twists that a four–dimensional hypercubist can generate. For instance, he can generate a pair of twists such as

$$(1234)_{abc}(1283)_{abd}$$

which to a Rubik cubist in the space of Face 4 appears as an isolated corner twist (a clockwise twist of the URF cubie); for he does not know that in another 3–space parallel to his, there is another Rubik cube (Face 8) on which a counterclockwise twist has appeared simultaneously.

The hypercubist can play many mystifying tricks on the poor cubist. When the latter has made an ordinary pair of twists, such as $(123)_{abc}(253)_{acb}$ - which are really $(1234)_{abc}(2534)_{acb}$ — the invisible hypercubist can snatch one of them away by conjugating with $R_{52}^2$, which moves $(2534)_{acb}$ to $(2578)_{acb}$ in Face 8. He can flip a single edge-cubie, and even stranger, perhaps, he can reflect a single corner-cubie, by $(1234)_{(ab)(cd)}$.

Analogously, the quarks may be shadows of hyperquarks, like the shadows of the puppets on the wall of Plato's cave, that can be produced or snatched away by an invisible puppeteer*. Some physicists believe that "wormholes" exist connecting "two distinct but asymptotically flat universes" and perhaps that particles can pass through such wormholes from one universe to the other (11). Perhaps this is the way to isolate a quark, like the hypercubist moves a twist from one face to another.

Finally, we would note that we have three types of tetrads, $N$, $S$, and $Z$, corresponding to the three colors of quarks, and each type has four "flavors". If the physicists had stopped when they had four flavors of quarks we would perhaps have a better analogy; but now that they believe in five or maybe six flavors (8) we find we have reached or surpassed the limits of our analogical powers.

* Behold! human beings living in an underground den,. . . here they have been since their childhood, . . . chained so that they cannot move, and can only see before them. . . . Behind them a fire is blazing at a distance, and between the fire and the prisoners there is a raised way; and . . . a low wall . . . like the screen which marionette players [use to] show their puppets. . .

You have shown me a strange image, and they are strange prisoners.

Like ourselves, I replied; and they see only their own shadows, or the shadows of one another, which the fire throws on the opposite wall of the cave?. . . And of the objects which are being carried [by the puppeteers] in like manner they would only see the shadows?

Yes, he said. . .

To them, I said, the truth would be literally nothing but the shadows of the images.

– Plato, The Republic, Book VII

Assuredly not, he said; I have hardly ever known a mathematician who was capable of reasoning.

– op. cit. sup.

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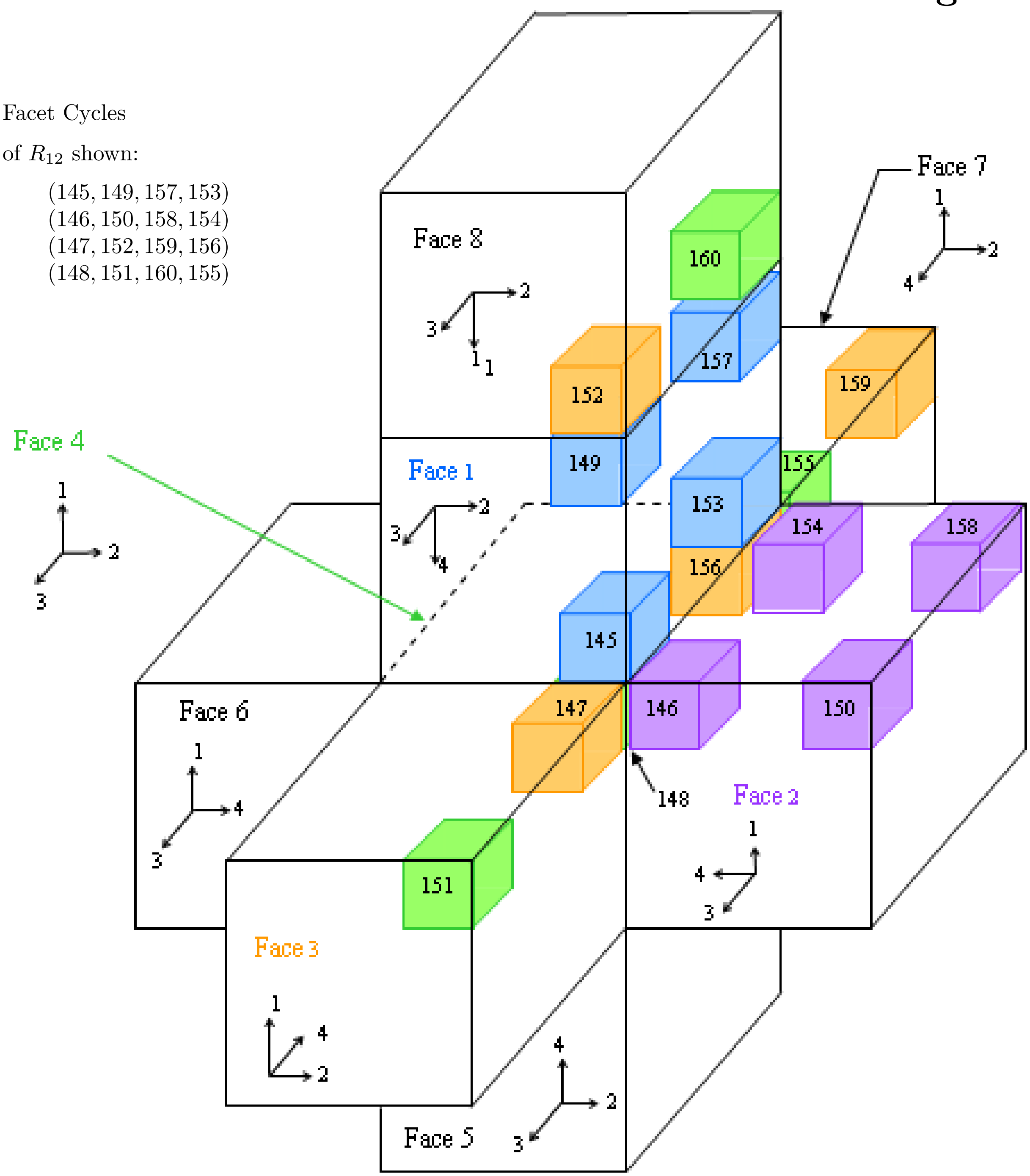
**Figure 1**

Facet Cycles

of $R_{12}$ shown:

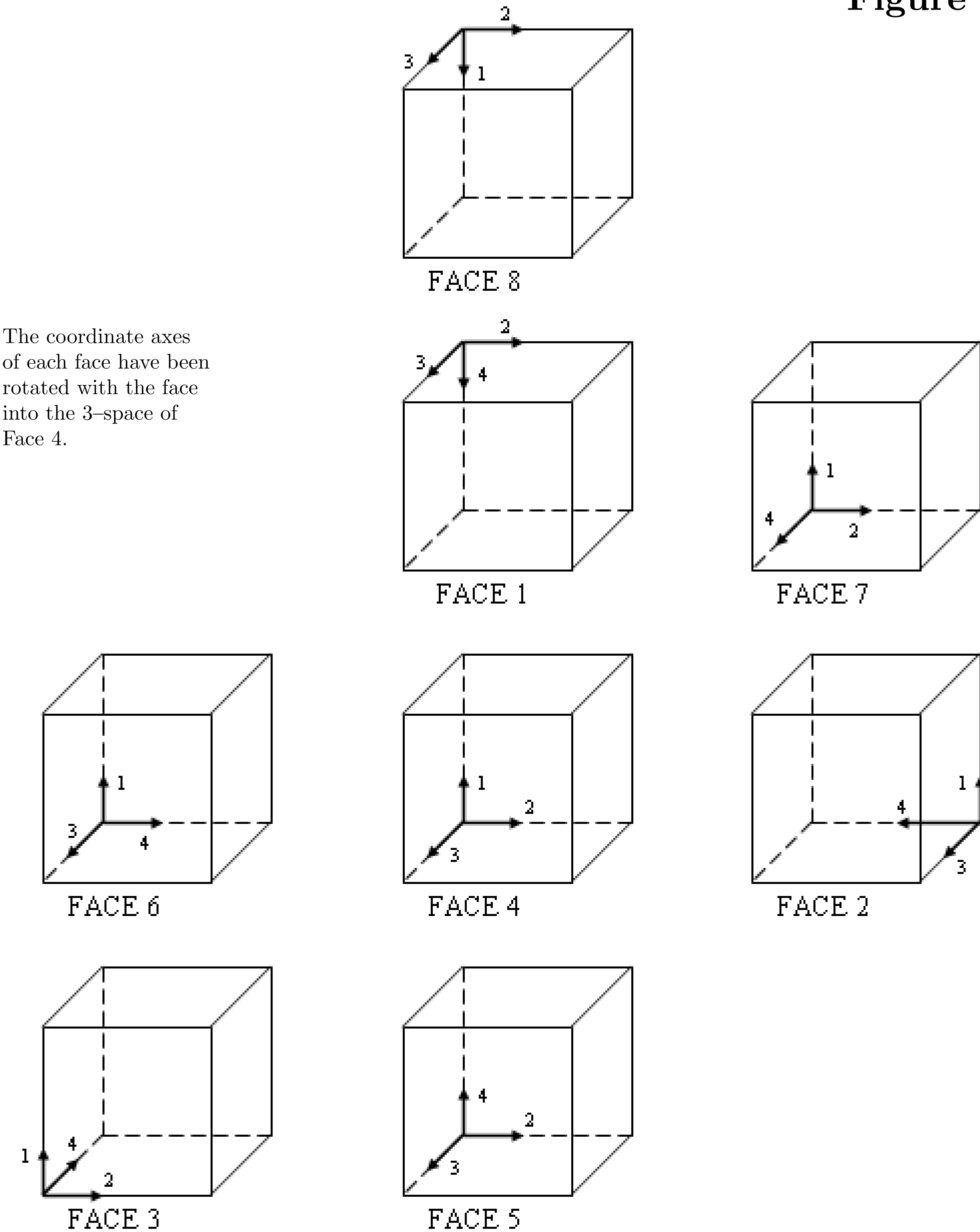
$(145, 149, 157, 153)$
$(146, 150, 158, 154)$
$(147, 152, 159, 156)$
$(148, 151, 160, 155)$




**RUBIK TESSERACT**

The large cubes are faces of the tesseract that have been rotated into the 3–space of Face 4, which is hidden behind Face 3 and below Face 1. The small cubes are the facets of a tetrad as it moves through the cycle $(1234, 1283, 1278, 1247)$ under the action of rotation $R_{12}$. The "start" position of this tetrad, as indicated by the colors of its facets, is 1234.

**Figure 2**




The coordinate axes of each face have been rotated with the face into the 3–space of Face 4.

## “Pin–Board” Diagram of Tesseract

**TABLE 1 – BASIC ROTATIONS**

$$R_{12} = (13,18,17,14)(123,128,127,124)(134,183,178,147)(163,168,167,164)$$
$$(1234,1283,1278,1247)(1634,1683,1678,1647)$$

$$R_{13} = (12,14,16,18)(123,143,163,183)(124,146,168,182)(127,147,167,187)$$
$$(1234,1436,1638,1832)(1274,1476,1678,1872)$$

$$R_{14} = (12,17,16,13)(123,172,167,136)(124,174,164,134)(128,178,168,138)$$
$$(1234,1724,1674,1364)(1238,1728,1678,1368)$$

$$R_{21} = (23,24,27,28)(123,124,127,128)(523,524,527,528)(234,247,278,283)$$
$$(1234,1247,1278,1283)(5234,5247,5278,5283)$$

$$R_{23} = (12,82,52,42)(123,823,523,423)(124,821,528,425)(127,827,527,427)$$
$$(1234,8231,5238,4235)(1274,8271,5278,4275)$$

$$R_{24} = (12,32,52,72)(123,325,527,721)(124,324,524,724)(128,328,528,728)$$
$$(1234,3254,5274,7214)(1238,3258,5278,7218)$$

$$R_{31} = (23,83,63,43)(123,183,163,143)(523,583,563,543)(234,832,638,436)$$
$$(1234,1832,1638,1436)(5234,5832,5638,5436)$$

$$R_{32} = (13,43,53,83)(123,423,523,823)(134,435,538,831)(163,463,563,863)$$
$$(1234,4235,5238,8231)(1634,4635,5638,8631)$$

$$R_{34} = (13,63,53,23)(123,613,563,253)(134,634,534,234)(138,638,538,238)$$
$$(1234,6134,5634,2534)(1238,6138,5638,2538)$$

$$R_{41} = (24,34,64,74)(124,134,164,174)(524,534,564,574)(234,364,674,724)$$
$$(1234,1364,1674,1724)(5234,5364,5674,5724)$$

$$R_{42} = (14,74,54,34)(124,724,524,324)(134,714,574,354)(164,764,564,364)$$
$$(1234,7214,5274,3254)(1634,7614,5674,3654)$$

$$R_{43} = (14,24,54,64)(124,254,564,614)(134,234,534,634)(174,274,574,674)$$
$$(1234,2534,5634,6134)(1274,2574,5674,6174)$$

$$R_{52} = (53,54,57,58)(523,524,527,528)(534,547,578,583)(563,564,567,568)$$
$$(5234,5247,5278,5283)(5634,5647,5678,5683)$$

$$R_{53} = (52,58,56,54)(523,583,563,543)(524,582,568,546)(527,587,567,547)$$
$$(5234,5832,5638,5436)(5274,5872,5678,5476)$$

$$R_{54} = (52,53,56,57)(523,536,567,572)(524,534,564,574)(528,538,568,578)$$
$$(5234,5364,5674,5724)(5238,5368,5678,5728)$$

$$R_{61} = (63,68,67,64)(163,168,167,164)(563,568,567,564)(634,683,678,647)$$
$$(1634,1683,1678,1647)(5634,5683,5678,5647)$$

$$R_{63} = (16,46,56,86)(163,463,563,863)(164,465,568,861)(167,467,567,867)$$
$$(1634,4635,5638,8631)(1674,4675,5678,8671)$$

$$R_{64} = (16,76,56,36)(163,761,567,365)(164,764,564,364)(168,768,568,368)$$
$$(1634,7614,5674,3654)(1638,7618,5678,3658)$$

**TABLE 1 - BASIC ROTATIONS**(continued)

$$R_{71} = (27,47,67,87)(127,147,167,187)(527,547,567,587)(274,476,678,872)$$
$$(1274,1476,1678,1872)(5274,5476,5678,5872)$$

$$R_{72} = (17,87,57,47)(127,827,527,427)(167,867,567,467)(174,871,578,475)$$
$$(1274,8271,5278,4275)(1674,8671,5678,4675)$$

$$R_{74} = (17,27,57,67)(127,257,567,617)(174,274,574,674)(178,278,578,678)$$
$$(1274,2574,5674,6174)(1278,2578,5678,6178)$$

$$R_{81} = (28,78,68,38)(128,178,168,138)(528,578,568,538)(238,728,678,368)$$
$$(1238,1728,1678,1368)(5238,5728,5678,5368)$$

$$R_{82} = (18,38,58,78)(128,328,528,728)(138,358,578,718)(168,368,568,768)$$
$$(1238,3258,5278,7218)(1638,3658,5678,7618)$$

$$R_{83} = (18,68,58,28)(128,618,568,258)(138,638,538,238)(178,678,578,278)$$
$$(1238,6138,5638,2538)(1278,6178,5678,2578)$$

## Table 2 — Rubik N–topes

| Dimension $= N$ | | 3 | | 4 | | 5 | |
|---|---|---|---|---|---|---|---|
| Number of Faces $= a_{N,1} = 2N$ | | 6 | | 8 | | 10 | |
| Rotation "Axes" per Face $= (1/2)(N-1)(N-2)$ | | 1 | | 3 | | 6 | |
| Number of Basic Rotations $= N(N-1)(N-2)$ | | 6 | | 24 | | 80 | |
| Number of Movable "Topies" $= 3^N - 2N - 1$ | | 24 | | 72 | | 232 | |
| **Number of n-ads and facets** | | $a_{3,n}$ | $f_{3,n}$ | $a_{4,n}$ | $f_{4,n}$ | $a_{5,n}$ | $f_{5,n}$ |
| | $n=2$ | 12 | 24 | 24 | 48 | 40 | 80 |
| n-ads : $a_{N,n} = 2^n \dfrac{N!}{(N-n)!\,n!}$ | $n=3$ | 8 | 24 | 32 | 96 | 80 | 240 |
| | $n=4$ | | | 16 | 64 | 80 | 320 |
| facets : $f_{N,n} = n\,a_{N,n}$ | $n=5$ | | | | | 32 | 160 |
| | Total | 20 | 48 | 72 | 208 | 232 | 800 |
| **Number of n-ads per face** | | | | | | | |
| | $n=2$ | 4 | | 6 | | 8 | |
| $b_{N,n} = a_{N-1,n-1} = 2^{n-1}\dfrac{(N-1)!}{(N-n)!(n-1)!}$ | $n=3$ | 4 | | 12 | | 24 | |
| | $n=4$ | | | 8 | | 32 | |
| | $n=5$ | | | | | 16 | |
| **Number of n-ads per face that do not move in a specific rotation of that face** | $n=2$ | 0 | | 0 | | 4 | |
| $c_{N,n} = \dfrac{2^{n-1}}{(n-1)!}(N-3)\dots(N-n-1)$ | $n=3$ | 0 | | 0 | | 4 | |
| $c_{N,n} = 0 \quad$ if $\quad n = N \quad$ or $\quad n = N-1$ | $n=4$ | | | 0 | | 0 | |
| | $n=5$ | | | | | 0 | |
| **Number of 4–cycles per rotation** | | | | | | | |
| $d_{N,n} = (1/4)(b_{N,n} - c_{N,n})$ | $n=2$ | 1 | | 1 | | 1 | |
| $d_{N,2} = 1$ | $n=3$ | 1 | | 3 | | 5 | |
| $d_{N,3} = 2N-5$ | $n=4$ | | | 2 | | 8 | |
| $d_{N,n}$ is even for $n > 3$ | $n=5$ | | | | | 4 | |

## APPENDIX A
## Useful Moves on the Rubik Tesseract

The basic rotations $R_{ij}$ are listed in Table 1. Useful moves are designated $Q_1$, $Q_2$, etc. The subscripts on the Q's have no significance as coordinates. Other moves needed temporarily are denoted by other capital letters. Moves are written in order from *left* to *right*; e.g., $R_{12}R_{13}$ means that $R_{12}$ is performed first, followed by $R_{13}$. The following notation is used for brevity:

Inverse : The inverses of $R_{ij}$, $Q_i$, etc. are $R'_{ij}$, $Q'_i$, etc.

Slice : $(R_{ij})_s = R_{ij}R'_{-ij}$

Antislice : $(R_{ij})_a = R_{ij}R_{-ij}$

Commutator : $(XY)_c = XYX'Y'$

Transpose : $Q[ij]_k$ means that subscripts $i$ and $j$ are interchanged on all the rotations that comprise $Q_k$, and that all the rotations are inverted. The effect of this is to transpose $i$ and $j$ in the permutation produced by $Q_k$.

For example if

$$Q_k = R_{52}R_{34} = (53, 54, 57, 58, 23, 13, 63) \text{ etc.},$$

then

$$Q[12]_k = R'_{61}R\prime_{34} = (63, 64, 67, 68, 13, 23, 53) \text{ etc.}$$

Note that in the permutation, 53 is treated as $1'3$ which changes to $2'3 = 63$ etc. Similarly, in the subscripts on the R's, $52 = -12$ changes to $-21$ or 61, etc. Note also that the rotations are inverted even if their subscripts are not involved in the transposition, as $R_{34}$ above. Transposition is a way of turning the tesseract around in 4-space.

### Simple Moves

We began looking for useful moves by following D. Singmaster's advice for the Rubik cube, looking at some simple moves. Almost at once we found two which led immediately to our first useful move, $Q_1$. These were

$$\begin{aligned}(R_{12})_s R_{21}^2 (R_{12})'_s =& (23, 27)(24, 28)(123, 127)(124, 128)\\ & (523, 527)(524, 528)(234, 278)(238, 274)\\ & (1234, 1278)(1238, 1274)(5234, 5278)(5238, 5274)\end{aligned}$$

and

$$\begin{aligned}(R_{12}^2 R_{23}^2)^2 =& (123, 127)(124, 128)(523, 527)(524, 528)\\ & (1234, 1278)(1238, 1274)(5234, 5278)(5238, 5274)\end{aligned}$$

Neither of these moves is very useful in itself, but they are obviously alike in most cycles, so we take their product:

$$Q_1 = (R_{12})_s R_{21}^2 (R_{12})'_s (R_{12}^2 R_{23}^2)^2 = (23, 27)(24, 28)(234, 278)(238, 274)$$

$Q_1$ is an interchange of two pairs of dyads and two pairs of triads. It turns out that from $Q_1$ we can generate a series of useful moves, including all those necessary to prove the order of the group of the Rubik tesseract. Some simple moves related to $Q_1$ are:

$$Q_2 = (Q_1 R_{32})_c = (123, 827, 423)(523, 427, 823)$$

is a pair of 3-cycles of triads.

Then, letting $\mathrm{X} = (\mathrm{R}_{32})_\mathrm{a}\mathrm{R}_{14}(\mathrm{R}_{32})'_\mathrm{a}$ we obtain a single 3-cycle of triads:

$$\mathrm{Q}_3 = (\mathrm{Q}_1\mathrm{X})_\mathrm{c} = (238, 274, 638)$$

**Dyadic and Triadic Swaps**

Now we reduce $\mathrm{Q}_1$ to a pair of triadic interchanges by the following strategy. The conjugating factor $\mathrm{X} = (\mathrm{R}_{32})_\mathrm{a}\mathrm{R}^2_{14}(\mathrm{R}_{32})'_\mathrm{a}$ interchanges 238 and 274 with 674 and 638, which are not affected by $\mathrm{Q}_1$, being on the opposite side of the tesseract. So $\mathrm{X}\mathrm{Q}_1\mathrm{X}' = (23, 27)(24, 28)(234, 278)(638, 674)$ and (since $\mathrm{X}' = \mathrm{X}$):

$$\mathrm{Q}_4 = \mathrm{Q}_1\mathrm{X}\mathrm{Q}_1\mathrm{X}' = [\mathrm{Q}_1(\mathrm{R}_{32})_\mathrm{a}\mathrm{R}^2_{14}(\mathrm{R}_{32})'_\mathrm{a}]^2 = (238, 274)(638, 674)$$

This is the minimum number of triadic interchanges, since a single interchange would be an odd permutation. Next we reduce $\mathrm{Q}_1$ to a pair of dyadic interchanges, which is the minimum possible for the same reason. Let

$$\mathrm{Q}_5 = \mathrm{Q}[24]_1 = (43, 47)(42, 46)(432, 476)(436, 472)$$

Then

$$\mathrm{Q}_1\mathrm{Q}_5 = (23, 27)(43, 47)(24, 28, 64)(234, 278, 674)(238, 634, 274)$$

We eliminate the 3–cycles by cubing. Thus

$$\mathrm{Q}_6 = (\mathrm{Q}_1\mathrm{Q}_5)^3 = (23, 27)(43, 47)$$

**Tetradic Swaps**

The process for obtaining a pair of tetradic interchanges is longer, but it is fairly straightforward because we can treat it as a problem on the Rubik cube. If we think of Face 2 as a Rubik cube, we see that the effect of $\mathrm{Q}_1$ is to rotate the entire middle layer of the cube by 180 degrees around axis 1. Similarly, the effect of $\mathrm{R}^2_{21}$ is to rotate the entire cube 180 degrees around the same axis. Therefore $\mathrm{R}^2_{21}\mathrm{Q}_1$ has the effect of rotating the top and bottom layers 180 degrees without affecting the middle layer. That is, $\mathrm{R}^2_{21}\mathrm{Q}_1$ interchanges the opposite edge cubies and the diagonally opposite corner cubies, top and bottom. To cancel out the edge-cubie interchanges we need triadic interchanges involving Face 2. We get these by conjugations on $\mathrm{Q}_4$, as follows:

$$\begin{aligned}
\mathrm{Q}_7 &= \mathrm{R}^2_{42}\mathrm{Q}_4\mathrm{R}^2_{42} &&= (234, 238)(364, 368)\\
\mathrm{Q}_8 &= \mathrm{R}_{24}\mathrm{Q}_7\mathrm{R}'_{24} &&= (124, 128)(364, 368)\\
\mathrm{Q}_9 &= \mathrm{R}_{12}\mathrm{Q}_8\mathrm{R}'_{12} &&= (123, 127)(364, 368)\\
\mathrm{Q}_{10} &= \mathrm{R}'_{24}\mathrm{Q}_7\mathrm{R}_{24} &&= (524, 528)(364, 368)\\
\mathrm{Q}_{11} &= \mathrm{R}_{52}\mathrm{Q}_{10}\mathrm{R}'_{52} &&= (523, 527)(364, 368)\\
\mathrm{Q}_{12} &= \mathrm{Q}_8\mathrm{Q}_9 &&= (123, 127)(124, 128)\\
\mathrm{Q}_{13} &= \mathrm{Q}_{10}\mathrm{Q}_{11} &&= (523, 527)(524, 528)
\end{aligned}$$

$\mathrm{Q}_{12}$ and $\mathrm{Q}_{13}$ are the desired moves which have the effect in Face 2 of swapping the opposite edge-cubies on the top and bottom. So

$$\mathrm{Q}_{14} = (\mathrm{R}^2_{21}\mathrm{Q}_1)\mathrm{Q}_{12}\mathrm{Q}_{13} = (1234, 1278)(1247, 1283)(5234, 5278)(5247, 5283)$$

We now have, in $Q_{14}$, an interchange of four pairs of tetrads. We next set out to reduce this to two pairs by the same strategy that we got $Q_4$, but it is difficult to conjugate one of these pairs of tetrads to the other side of the tesseract without affecting any of the others.

We first rearrange the interchanging pairs, as follows:

$$\mathrm{Q}_{15} = \mathrm{R}'_{32}\mathrm{R}^2_{52}\mathrm{R}'_{42}\mathrm{Q}_{14}\mathrm{R}_{42}\mathrm{R}^2_{52}\mathrm{R}_{32} \ = (1234, 1283)(1278, 7214)(5234, 5283)(5278, 7254)$$

We remark that the two tetrads, 1247 and 5247, have been re-oriented. We now move two pairs to the opposite side of the tesseract:

$$\mathrm{Q}_{16} = \mathrm{R}^2_{13}\mathrm{Q}_{15}\mathrm{R}^2_{13} = (1346, 1836)(1647, 7681)(5234, 5283)(5278, 7254)$$

(Actually, this move was unnecessary. We could have proceeded to the next step from $Q_{15}$.) We apply the commutator move:

$$\mathrm{Q}_{17} = (\mathrm{Q}_{16}\mathrm{R}'_{64})_{\mathrm{c}} = (1346, 1836)(5678, 7654)(1876)_{\mathrm{abc}}(1746)_{\mathrm{acb}}$$

By cubing $Q_{17}$ we have, finally, two pairs of tetrads:

$$\mathrm{Q}^3_{17} = (1346, 1836)(5678, 7654)$$

Now we have found two pairs of interchanges on dyads ($Q_6$), on triads ($Q_4$), and on tetrads ($Q^3_{17}$). By conjugation we can obtain such transpositions on any two pairs of dyads, triads, or tetrads. Since any permutation can be expressed as a product of transpositions, we can build up any positional permutation that is even for dyads, triads, and tetrads. In the case of a permutation that is odd for dyads and triads (and necessarily even for tetrads) we can multiply it by any basic rotation and the product will be even for dyads, triads, and tetrads, as illustrated below.

**Dyad-Triad Pair Swap**

To obtain an interchange of one pair of dyads and one pair of triads, as, for example:

$$\mathrm{Q}_{36} = (23, 27)(234, 278)$$

we proceed as follows. First we find

$$\mathrm{Q}_{36}\mathrm{R}_{21} = \mathrm{A\,B\,C}$$

where

$$\begin{aligned}
\mathrm{A} &= (23, 28)(24, 27)(234, 283)(247, 278)\\
\mathrm{B} &= (123, 124, 127, 128)(523, 524, 527, 528)\\
\mathrm{C} &= (1234, 1247, 1278, 1283)(5234, 5247, 5278, 5283)
\end{aligned}$$

We can build up A from conjugations of $Q_6$ and $Q_{12}$, i.e.:

$$\begin{aligned}
\mathrm{A}_1 &= \ \mathrm{R}_{34}\mathrm{R}'_{21}\mathrm{R}'_{34}\mathrm{Q}_6\mathrm{R}_{34}\mathrm{R}_{21}\mathrm{R}'_{34} \ = (23, 28)(43, 47)\\
\mathrm{A}_2 &= \ \mathrm{R}_{74}\mathrm{R}'_{21}\mathrm{R}'_{74}\mathrm{Q}_6\mathrm{R}_{74}\mathrm{R}_{21}\mathrm{R}'_{74} \ = (24, 27)(43, 47)\\
\mathrm{A}_3 &= \mathrm{R}_{42}\mathrm{R}'_{12}\mathrm{R}_{32}\mathrm{Q}_{12}\mathrm{R}'_{32}\mathrm{R}_{12}\mathrm{R}'_{42} = (234, 283)(123, 127)\\
\mathrm{A}_4 &= \mathrm{R}_{82}\mathrm{R}'_{12}\mathrm{R}_{72}\mathrm{Q}_{12}\mathrm{R}'_{72}\mathrm{R}_{12}\mathrm{R}'_{82} = (247, 278)(123, 127)
\end{aligned}$$

So

$$\mathrm{A} = \mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3\,\mathrm{A}_4$$

Likewise, B can be built up from $Q_9$ and $Q_{11}$:

$$\begin{aligned}
B_2 &= Q_9 Q_{11} &&= (123,127)(523,527)\\
B_1 &= R_{32}R'_{12}R_{52}R'_{32}B_2R_{32}R'_{52}R_{12}R'_{32} &&= (123,124)(523,524)\\
B_3 &= R'_{32}R_{12}R'_{52}R_{32}B_2R'_{32}R_{52}R'_{12}R_{32} &&= (123,128)(523,528)
\end{aligned}$$

and

$$B = B_1\, B_2\, B_3$$

Then C can be built up from $Q_{17}^3$ :

$$\begin{aligned}
C_0 &= R'_{12}R^2_{14}Q^3_{17}R^2_{14}R_{12} &&= (1234,1247)(5678,7654)\\
C_1 &= C_0R^2_{24}C_0R^2_{24} &&= (1234,1247)(5234,5247)\\
C_2 &= R^2_{31}R'_{21}R^2_{31}C_1R^2_{31}R_{21}R^2_{31} &&= (1234,1278)(5234,5278)\\
C_3 &= R_{21}C_1R'_{21} &&= (1234,1283)(5234,5283)
\end{aligned}$$

and

$$C = C_1\, C_2\, C_3$$

Thus we obtain

$$Q_{36} = ABCR'_{21} = (23,27)(234,278)$$

The number of basic rotations to get $Q_{36}$ is 11,919.

In making the tetradic conjugations above it is helpful to note that the proper orientations can be produced most easily if the desired interchange changes the same number of faces as the original interchange. In the case above, for example, we wish to conjugate from $Q^3_{17}$ to $C_1$. Each interchange in $C_1$ holds two faces fixed. In $Q^3_{17}$, the interchange $(1346,1836)$ also holds two faces fixed (1 and 6) but the interchange $(5678,7654)$ changes three faces. Therefore we conjugate in two steps using only the first interchange of $Q^3_{17}$. That is, we first get $C_0$ from $(1346,1836)$, leaving $(5678,7654)$ alone. Then we get

$$R^2_{24}C_0R^2_{24} = (5234,5247)(5678,7654)$$

again leaving the second interchange alone. Then the product of these gives $C_1$.

**Tetradic Twists and Crosses**

As a bonus from $Q_{17}$ we have a pair of twists:

$$Q^2_{17} = (1876)_{\mathrm{acb}}(1746)_{\mathrm{abc}}$$

For convenience we move them to the other side of the tesseract:

$$Q_{18} = R^2_{13}R^2_{61}Q^2_{17}R^2_{61}R^2_{13} = (1234)_{\mathrm{acd}}(1283)_{\mathrm{adc}}$$

We now proceed to obtain the other permutations of orientation of tetrad 1234. We first make the same twist on a third tetrad:

$$Q_{19} = R_{82}Q_{18}R'_{82} = (1234)_{\mathrm{acd}}(1278)_{\mathrm{adc}}$$

Next we obtain a different twist by transposition:

$$Q[12]_{19} = (2134)_{\mathrm{acd}}(2178)_{\mathrm{adc}} = (1234)_{\mathrm{bcd}}(1278)_{\mathrm{bdc}}$$

Combining $Q_{18}$ and $Q[12]_{19}$ gives a cross on 1234:

$$\begin{aligned} Q_{18}Q[12]_{19} &= (1234)_{(bcd)(acd)}(1283)_{adc}(1278)_{bdc} \\ &= (1234)_{(ac)(bd)}(1283)_{adc}(1278)_{bdc} \end{aligned}$$

or

$$Q[12]_{19}Q_{18} = (1234)_{\mathrm{(ad)(bc)}}(1283)_{\mathrm{adc}}(1278)_{\mathrm{bdc}}$$

(Note that the permutations *(acd)* and *(bcd)* multiply in reverse order to the moves $Q_{18}$ and $Q[12]_{19}$.) So we get

$$\begin{aligned} Q_{20} &= (Q_{18}Q[12]_{19})^3 = (1234)_{(ac)(bd)} \\ Q_{21} &= (Q[12]_{19}Q_{18})^3 = (1234)_{(ad)(bc)} \end{aligned}$$

The third cross is

$$Q_{22} = Q_{20}Q_{21} = (1234)_{\mathrm{(ab)(cd)}}$$

We can also obtain $Q_{22}$ in half as many steps from:

$$Q_{22} = Q_{18}Q[12]_{19}(Q[12]_{19}Q_{18})' = (Q_{18}Q[12]_{19})_{\mathrm{c}}$$

By multiplying $Q_{18}$ in turn by each of the crosses we obtain all of the S-twists on tetrad 1234:

$$\begin{aligned} Q_{18}Q_{20} &= (1234)_{adb}(1283)_{adc} \\ Q_{18}Q_{21} &= (1234)_{bdc}(1283)_{adc} \\ Q_{18}Q_{22} &= (1234)_{abc}(1283)_{adc} \end{aligned}$$

and by squaring these moves (including $Q_{18}$) we get all the Z-twists on 1234. We can easily move the crosses to tetrad 1283 [$R'_{12}Q_{20}R_{12} = (1283)_{\mathrm{(ac)(bd)}}$, etc.] and generate all eight twists on that tetrad. In this way we can generate all of the 32 possible pairs of 1234 and 1283 with an S-twist on one of them and a Z-twist on the other. We can easily move the crosses and twist-pairs anywhere on the tesseract, and thus obtain all possible permutations of tetradic orientations.

**Dyadic Flips and Triadic Twists**

Dyads have only two orientations, and flips must occur in pairs. It is easy to find a representative pair by transposition. We start with

$$Q_6 = (23, 27)(43, 47)$$

Then

$$\begin{aligned} Q_{23} &= R_{14}R'_{72}Q_6R_{72}R'_{14} = (12, 43)(23, 27) \\ Q[12]_{23} &= (21, 43)(13, 17) \\ Q[12]_6 &= (13, 17)(43, 47) \end{aligned}$$

and so

$$Q_{24} = (Q_6Q[12]_6)(Q_{23}Q[12]_{23}) = (12, 43)(21, 43) = (12)_{\mathrm{ab}}(34)_{\mathrm{ab}}$$

A triadic twist can also be found by use of transpositions. These twists can occur singly. We start with $Q_7$, which we move to

$$Q_{25} = R_{12}R_{82}Q_7R'_{82}R'_{12} = (123, 324)(163, 364)$$

Transposing 2–3, we obtain

$$Q[23]_{25} = (132, 234)(172, 274) = (123, 243)(172, 274)$$

So

$$Q_{25}Q[23]_{25} = (123)_{\mathrm{abc}}(234)_{\mathrm{abc}}(163,364)(172,274)$$

and

$$Q_{26} = (Q_{25}Q[23]_{25})^2 = (123)_{\mathrm{acb}}(234)_{\mathrm{acb}}$$

To reduce this to a single twist we make another transposition:

$$\begin{aligned} Q[12]_{26} &= (123)_{\mathrm{abc}}(134)_{\mathrm{acb}} \\ Q_{27} &= R'_{13}Q_{26}R_{13} = (134)_{\mathrm{abc}}(234)_{\mathrm{acb}} \\ Q_{28} &= Q[12]_{26}Q_{27} = (123)_{\mathrm{abc}}(234)_{\mathrm{acb}} \end{aligned}$$

So

$$Q_{29} = Q_{26}Q'_{28} = Q_{26}Q^2_{28} = (123)_{\mathrm{abc}}$$

The reverse twist is of course $Q^2_{29} = (123)_{\mathrm{acb}}$.

These triadic twists and the dyadic flip-pair can obviously be moved anywhere on the tesseract.

**Triadic Reflections — "Gene Splicing"**

The final type of permutation of orientation we need to find is a pair of triadic reflections (they cannot occur singly), which we will do by a method we call "gene splicing" for reasons that will be obvious. We first look for a move that includes such a reflection as part of the permutation. Such moves are easy to find. For instance

$$Q_{30} = R_{31}R'_{12}R_{41} = (134)_{\mathrm{bc}}X$$

The remainder of the permutation, denoted by X, is a long and complicated sequence of cycles which involve all but seven of the 31 other triads. We call $Q_{30}$ the "chromosome". We do not need to concern ourselves with the details of X except to identify the "neutral" triads that are not affected by it, some of which we wish to use for conjugating purposes. The four we use are 275, 872, 576, and 678, all of which have facets in Face 7. We start with $Q_4 = (238,274)(638,674)$ and convert it to

$$A = R^2_{82}Q_4R^2_{82} = (274,278)(674,678)$$

in which all the triads also have facets in Face 7. We can therefore move A to the four neutral triads by a Rubik-cube move; for instance

$$Q_{31} = R'_{53}R^2_{43}R^2_{53}R_{43}AR'_{\ 43}R^2_{53}R^2_{43}R_{53} \ = (275,872)(576,678)$$

[That is, if we regard Face 7 as a Rubik cube, A is a pair of edge-swaps : Right-Front with Right-Back and Left-Front with Left-Back; while $Q_{31}$ swaps Right-Down with Right-Back and Left-Down with Left-Back. This conjugation is easily found on the Rubik cube.]

The next step is to move 134 to one of the neutral positions without affecting the other three positions:

$$\begin{aligned} Q_{32} &= R^2_{14}R'_{13}R_{53}R_{23}R'_{53}Q_{31}R_{53}R'_{23}R'_{53}R_{13}R^2_{14} \\ &= (275,134)(576,678) \end{aligned}$$

The third step is to twist triad 275, which we do by conjugating $Q_{29}$:

$$Q_{33} = R^2_{24}Q_{29}R^2_{24} = (275)_{\mathrm{abc}}$$

Step 4 is to splice this gene into the chromosome:

$$\mathrm{Q}_{33}\mathrm{Q}_{30} = (275)_{\mathrm{abc}}(134)_{\mathrm{bc}}\mathrm{X}$$

Step 5:

$\mathrm{Q}_{32}$ is a "reagent" that will react with the chromosome $\mathrm{Q}_{33}\mathrm{Q}_{30}$ only at the "gene sites" 275 and 134. Carrying out this reaction, we *mutate* gene 134 from type bc to type abc and we mutate gene 275 in the reverse manner:

$$\mathrm{Q}_{34} = \mathrm{Q}_{32}(\mathrm{Q}_{33}\mathrm{Q}_{30})\mathrm{Q}_{32} = (275)_{\mathrm{bc}}(134)_{\mathrm{abc}}\mathrm{X}$$

Step 6:

Finally, mixing the two chromosomes $\mathrm{Q}_{30}$ and $\mathrm{Q}_{34}$, we isolate the genes 134 and 275:

$$\mathrm{Q}_{35} = \mathrm{Q}'_{30}\mathrm{Q}_{34} = (134)_{(\mathrm{abc})(\mathrm{bc})}(275)_{(\mathrm{bc})}\mathrm{X}'\mathrm{X} = (134)_{\mathrm{ac}}(275)_{\mathrm{bc}}$$

From $\mathrm{Q}_{35}$ and $\mathrm{Q}_{33}$ we can get the other two reflections on triad 275; that is:

$$\mathrm{Q}_{33}\mathrm{Q}_{35} = (134)_{ac}(275)_{ab}$$
$$\mathrm{Q}_{35}\mathrm{Q}_{33} = (134)_{ac}(275)_{ac}$$

Similarly we can generate the other two reflections on 134. Thus all nine combinations of reflections of 134 and 275 can be found, and we can move these anywhere on the tesseract.

In summary, we have shown how to generate any permutation of the Rubik tesseract group.

```
PROGRAM TESS2X (TDATA); {Search for Useful Moves on Rubik Tesseract}
CONST CR=13;
      ORG=%C000;
      ORG2=%C800;
      TRUE=1;
      FALSE=0;
VAR FACE:INTEGER;
    M, FLAG   :ARRAY[208] OF INTEGER;
    OK, DONE, RECAP, FACE, ANS, NMOVES : INTEGER;
PROC CRLF;  BEGIN WRITE(13,10) END;
PROC INITIALIZE;
   VAR J,K,L1,L2,L3:INTEGER;
   BEGIN
      NMOVES := 0;
      FOR J:=1 TO 208 DO
         BEGIN
            M[J]:=J;
            FLAG[J] := TRUE;
            READ(TDATA,K);
            MEM[ORG2+J] := K
         END;
      FOR J:=1 TO 1824 DO
         BEGIN
            READ(TDATA,L1,L2,L3);
            MEM[ORG+J-1]:=100*L1+10*L2+L3-5328
         END
   END;
PROC REFORM;
   VAR J:INTEGER;
   BEGIN
      NMOVES := 0;
      FOR J:=1 TO 208 DO
         BEGIN
            M[J]:=J;
            FLAG[J] := TRUE
         END
   END;
PROC Z(N);   {move proc}
   VAR START, S, J : INTEGER;
   PROC PERMUTE(K1,K2,K3,K4);
      VAR SAVE:INTEGER;
      BEGIN
         SAVE:=M[K1]; M[K1]:=M[K4]; M[K4]:=M[K3]; M[K3]:=M[K2]; M[K2]:=SAVE
      END;   {permute}
   BEGIN
      OK := TRUE;
      CASE N OF
    12: START := ORG;         -12,52: START := ORG + 912;
    13: START := ORG + 76;    -13,53: START := ORG + 988;
    14: START := ORG + 152;   -14,54: START := ORG + 1064;
    21: START := ORG + 228;   -21,61: START := ORG + 1140;
    23: START := ORG + 304;   -23,63: START := ORG + 1216;
    24: START := ORG + 380;   -24,64: START := ORG + 1292;
```

```
     31: START := ORG + 456;   -31,71: START := ORG + 1368;
     32: START := ORG + 532;   -32,72: START := ORG + 1444;
     34: START := ORG + 608;   -34,74: START := ORG + 1520;
     41: START := ORG + 684;   -41,81: START := ORG + 1596;
     42: START := ORG + 760;   -42,82: START := ORG + 1672;
     43: START := ORG + 836;   -43,83: START := ORG + 1748;
     99: RECAP := TRUE;
    999: REFORM
      ELSE
         OK := FALSE
      END;   {case}
   IF NOT OK THEN
   BEGIN
IF (N>100) AND (N<199) THEN
   BEGIN
   OK:= TRUE;
   IF N=101 THEN
        BEGIN
         Z(12);
         Z(-12);    Z(-12);    Z(-12);
         Z(21);     Z(21);
         Z(-12);
         Z(12);     Z(12);     Z(12);
         Z(12);     Z(12);
         Z(23);     Z(23);
         Z(12);     Z(12);
         Z(23);     Z(23);
        END
   ELSE IF N=102 THEN
        BEGIN
         Z(101);
         Z(32);
         Z(101);
         Z(32); Z(32); Z(32)
        END
   ELSE IF N=103 THEN
        BEGIN
         Z(101);
         Z(32);     Z(-32);
         Z(14);
         Z(-32);    Z(-32);    Z(-32);
         Z(32);     Z(32);     Z(32);
         Z(101);
         Z(32);     Z(-32);
         Z(14);     Z(14);     Z(14);
         Z(-32);    Z(-32);    Z(-32);
         Z(32);     Z(32);     Z(32)
        END
   ELSE IF N=104 THEN
        BEGIN
         Z(101);
         Z(32);     Z(-32);
         Z(14);     Z(14);
```

```
        Z(-32);    Z(-32);    Z(-32);
        Z(32);     Z(32);     Z(32);
        Z(101);
        Z(32);     Z(-32);
        Z(14);     Z(14);
        Z(-32);    Z(-32);    Z(-32);
        Z(32);     Z(32);     Z(32)
       END
  ELSE IF N=105 THEN
       BEGIN
        Z(14);
        Z(-14);    Z(-14);    Z(-14);
        Z(41);     Z(41);
        Z(-14);
        Z(14);     Z(14);     Z(14);
        Z(14);     Z(14);
        Z(43);     Z(43);
        Z(14);     Z(14);
        Z(43);     Z(43);
       END
  ELSE IF N=106 THEN
       BEGIN
        Z(101); Z(105);
        Z(101); Z(105);
        Z(101); Z(105)
       END
  ELSE IF N=107 THEN
        BEGIN
         Z(42); Z(42); Z(104); Z(42); Z(42)
        END
  ELSE IF N=108 THEN
        BEGIN
         Z(24); Z(107); Z(24); Z(24); Z(24)
        END
  ELSE IF N=109 THEN
        BEGIN
         Z(12); Z(108); Z(12); Z(12); Z(12)
        END
  ELSE IF N=110 THEN
        BEGIN
         Z(24); Z(24); Z(24); Z(107); Z(24)
        END
  ELSE IF N=111 THEN
        BEGIN
         Z(52); Z(110); Z(52); Z(52); Z(52)
        END
  ELSE IF N=112 THEN
        BEGIN
         Z(108); Z(109)
        END
  ELSE IF N=113 THEN
        BEGIN
         Z(110); Z(111)
```

```
      END
ELSE IF N=114 THEN
      BEGIN
       Z(21); Z(21);
       Z(101);
       Z(24); Z(42); Z(42);
       Z(104);
       Z(42); Z(42); Z(24); Z(24); Z(24);
       Z(12); Z(24); Z(42); Z(42);
       Z(104);
       Z(42); Z(42); Z(24); Z(24); Z(24);
       Z(12); Z(12);Z(12);
       Z(24); Z(24); Z(24); Z(42); Z(42);
       Z(104);
       Z(42); Z(42); Z(24);
       Z(-12); Z(24); Z(24); Z(24); Z(42); Z(42);
       Z(104);
       Z(42); Z(42); Z(24); Z(-12); Z(-12); Z(-12)
      END
ELSE IF N=116 THEN
      BEGIN
       Z(13); Z(13); Z(32); Z(32); Z(32);
       Z(-12); Z(-12); Z(42); Z(42); Z(42);
       Z(114);
       Z(42); Z(-12); Z(-12); Z(32); Z(13); Z(13)
      END
ELSE IF N=117 THEN
      BEGIN
       Z(116); Z(-24); Z(-24); Z(-24);
       Z(116); Z(-24)
      END
ELSE IF N=118 THEN
      BEGIN
       Z(13); Z(13); Z(-21); Z(-21); Z(117); Z(117);
       Z(-21); Z(-21); Z(13); Z(13)
      END
ELSE IF N=119 THEN
      BEGIN
       Z(-42); Z(118); Z(-42); Z(-42); Z(-42)
      END
ELSE IF N=120 THEN
      BEGIN
       Z(118); Z(219);
       Z(118); Z(219);
       Z(118); Z(219)
      END
ELSE IF N=121 THEN
      BEGIN
       Z(219); Z(118); Z(219); Z(118); Z(219); Z(118)
       END
ELSE IF N=122 THEN
      BEGIN
       Z(120); Z(121)
```

```
          END
    ELSE IF N=123 THEN
         BEGIN
          Z(14); Z(-32); Z(-32); Z(-32); Z(106); Z(-32); Z(14); Z(14); Z(14)
         END
    ELSE IF N=124 THEN
         BEGIN
          Z(123); Z(223); Z(106); Z(206)
         END
    ELSE IF N=125 THEN
         BEGIN
          Z(12); Z(82); Z(107); Z(82); Z(82); Z(82); Z(12); Z(12); Z(12)
         END
    ELSE IF N=126 THEN
         BEGIN
          Z(125); Z(325); Z(125); Z(325)
         END
    ELSE IF N=127 THEN
         BEGIN
          Z(13); Z(13); Z(13); Z(126); Z(13)
         END
    ELSE IF N=128 THEN
         BEGIN
          Z(226); Z(127)
         END
    ELSE IF N=129 THEN
         BEGIN
          Z(126); Z(128); Z(128)
         END
    ELSE IF N=130 THEN
         BEGIN
          Z(31); Z(12); Z(12); Z(12); Z(41)
         END
    ELSE IF N=131 THEN
         BEGIN
          Z(53); Z(53); Z(53); Z(43); Z(43); Z(53); Z(53); Z(43);
          Z(82); Z(82); Z(104); Z(82); Z(82);
          Z(43); Z(43); Z(43); Z(53); Z(53); Z(43); Z(43); Z(53)
         END
    ELSE IF N=132 THEN
         BEGIN
          Z(14); Z(14); Z(13); Z(13); Z(13); Z(53); Z(23);
          Z(53); Z(53); Z(53); Z(131); Z(53);
          Z(23); Z(23); Z(23); Z(53); Z(53); Z(53); Z(13); Z(14); Z(14)
         END
    ELSE IF N=133 THEN
         BEGIN
          Z(24); Z(24); Z(129); Z(24); Z(24)
         END
    ELSE IF N=134 THEN
         BEGIN
          Z(132); Z(133); Z(130); Z(132)
         END
```

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    ELSE IF N=135 THEN
          BEGIN
           Z(41); Z(41); Z(41); Z(12); Z(31); Z(31); Z(31); Z(134)
          END
    ELSE IF N=136 THEN
          BEGIN
           Z(34); Z(21); Z(21); Z(21); Z(34); Z(34); Z(34); Z(106);
           Z(34); Z(21); Z(34); Z(34); Z(34);
           Z(74); Z(21); Z(21); Z(21); Z(74); Z(74); Z(74); Z(106);
           Z(74); Z(21); Z(74); Z(74); Z(74);
           Z(42); Z(12); Z(12); Z(12); Z(32); Z(112); Z(32); Z(32); Z(32);
           Z(12); Z(42); Z(42); Z(42);
           Z(82); Z(12); Z(12); Z(12); Z(72); Z(112); Z(72); Z(72); Z(72);
           Z(12); Z(82); Z(82); Z(82);
           Z(32); Z(12); Z(12); Z(12); Z(52); Z(32); Z(32); Z(32); Z(109);
           Z(111); Z(32); Z(52); Z(52); Z(52); Z(12); Z(32); Z(32); Z(32);
           Z(109); Z(111);
           Z(32); Z(32); Z(32); Z(12); Z(52); Z(52); Z(52); Z(32); Z(109);
           Z(111); Z(32); Z(32); Z(32); Z(52); Z(12); Z(12); Z(12); Z(32);
           Z(140);
           Z(31); Z(31); Z(21); Z(21); Z(21); Z(31); Z(31); Z(140);
           Z(31); Z(31); Z(21); Z(31); Z(31);
           Z(21); Z(140); Z(21); Z(21); Z(21);
           Z(21); Z(21); Z(21)
          END
    ELSE IF N=140 THEN
          BEGIN
           Z(12); Z(12); Z(12); Z(14); Z(14); Z(117); Z(117); Z(117);
           Z(14); Z(14); Z(12); Z(24); Z(24);
           Z(12); Z(12); Z(12); Z(14); Z(14); Z(117); Z(117); Z(117);
           Z(14); Z(14); Z(12); Z(24); Z(24)
          END
    ELSE OK:=FALSE
       END;
IF (N>200) AND (N<299) THEN
    BEGIN
    OK:= TRUE;
    IF N=201 THEN
          BEGIN
           Z(21); Z(21); Z(21); Z(-21); Z(12); Z(12); Z(-21); Z(-21); Z(-21);
           Z(21);      Z(21); Z(21); Z(13); Z(13); Z(21); Z(21); Z(13); Z(13)
          END
    ELSE IF N=204 THEN
          BEGIN
           Z(201); Z(31); Z(31); Z(31); Z(-31); Z(-31); Z(-31);
           Z(24); Z(24); Z(-31); Z(31);
           Z(201); Z(31); Z(31); Z(31); Z(-31); Z(-31); Z(-31);
           Z(24); Z(24); Z(-31); Z(31)
          END
    ELSE IF N=205 THEN
          BEGIN
           Z(24); Z(24); Z(24); Z(-24); Z(-24); Z(-24); Z(42); Z(42);
           Z(24);Z(-24);Z(24);Z(24);Z(43); Z(43);Z(24);Z(24);Z(43);Z(43)
```

```
          END
    ELSE IF N=206 THEN
          BEGIN
           Z(201); Z(205); Z(201); Z(205); Z(201); Z(205)
          END
    ELSE IF N=207 THEN
          BEGIN
           Z(41); Z(41); Z(204); Z(41); Z(41)
          END
    ELSE IF N=208 THEN
          BEGIN
           Z(14); Z(14); Z(14); Z(207); Z(14)
          END
    ELSE IF N=209 THEN
          BEGIN
           Z(21); Z(21); Z(21); Z(208); Z(21)
          END
    ELSE IF N=210 THEN
          BEGIN
           Z(14); Z(207); Z(14); Z(14); Z(14)
          END
    ELSE IF N=211 THEN
          BEGIN
           Z(-21); Z(-21); Z(-21); Z(210); Z(-21)
          END
    ELSE IF N=214 THEN
          BEGIN
           Z(12); Z(12); Z(201); Z(208); Z(209); Z(210); Z(211)
          END
    ELSE IF N=215 THEN
          BEGIN
           Z(31); Z(-21); Z(-21); Z(41); Z(214);
           Z(41); Z(41); Z(41); Z(-21); Z(-21); Z(31); Z(31); Z(31)
          END
    ELSE IF N=216 THEN
          BEGIN
           Z(23); Z(23); Z(215); Z(23); Z(23)
          END
    ELSE IF N=217 THEN
          BEGIN
           Z(216); Z(-14); Z(216); Z(-14); Z(-14); Z(-14)
          END
    ELSE IF N=218 THEN
          BEGIN
           Z(23); Z(23); Z(-12); Z(-12); Z(217); Z(217);
           Z(-12); Z(-12); Z(23); Z(23)
          END
    ELSE IF N=219 THEN
          BEGIN
           Z(-41); Z(-41); Z(-41); Z(218); Z(-41)
          END
    ELSE IF N=223 THEN
          BEGIN
```

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          Z(24); Z(24); Z(24); Z(-31); Z(206); Z(-31); Z(-31); Z(-31); Z(24)
         END
   ELSE IF N=225 THEN
         BEGIN
          Z(21); Z(21); Z(21); Z(-41); Z(-41); Z(-41); Z(207); Z(-41); Z(21)
         END
   ELSE IF N=226 THEN
         BEGIN
          Z(225); Z(425); Z(225); Z(425)
         END
   ELSE OK:=FALSE
   END;
IF (N>300) AND (N<399) THEN
   BEGIN
   OK:=TRUE;
   IF N=301 THEN
         BEGIN
          Z(13); Z(13); Z(13); Z(-13); Z(31); Z(31); Z(-13); Z(-13); Z(-13);
          Z(13); Z(13); Z(13); Z(32); Z(32); Z(13); Z(13); Z(32); Z(32)
         END
   ELSE IF N=304 THEN
         BEGIN
          Z(301); Z(23); Z(23); Z(23); Z(-23); Z(-23); Z(-23); Z(14); Z(14);
          Z(23); Z(-23);
          Z(301); Z(23); Z(23); Z(23); Z(-23); Z(-23); Z(-23); Z(14); Z(14);
          Z(23); Z(-23)
         END
   ELSE IF N=307 THEN
         BEGIN
          Z(43); Z(43); Z(304); Z(43); Z(43)
         END
   ELSE IF N=325 THEN
         BEGIN
          Z(13); Z(13); Z(13); Z(83); Z(83); Z(83); Z(307); Z(83); Z(13)
         END
   ELSE OK:=FALSE
      END;
IF (N>400) AND (N<499) THEN
   BEGIN
   OK:=TRUE;
   IF N=401 THEN
         BEGIN
          Z(23); Z(-23); Z(-23); Z(-23); Z(32); Z(32); Z(-23); Z(23); Z(23);
          Z(23); Z(23); Z(23); Z(31); Z(31); Z(23); Z(23); Z(31); Z(31)
         END
   ELSE IF N=404 THEN
         BEGIN
          Z(401); Z(13); Z(-13); Z(24); Z(24); Z(13); Z(13); Z(13); Z(-13);
          Z(-13); Z(-13);
          Z(401); Z(13); Z(-13); Z(24); Z(24); Z(13); Z(13); Z(13); Z(-13);
          Z(-13); Z(-13)
         END
   ELSE IF N=407 THEN
```

```
         BEGIN
          Z(43); Z(43); Z(404); Z(43); Z(43)
         END
   ELSE IF N=425 THEN
         BEGIN
          Z(23); Z(-43); Z(407); Z(-43); Z(-43); Z(-43); Z(23); Z(23); Z(23)
         END
   ELSE OK:=FALSE
      END
   END;  {if not ok}
      IF (OK) AND (N<99) THEN
            BEGIN
               NMOVES := NMOVES + 1;
               FOR J:=1 TO 19 DO
                  BEGIN
                     S := START + 4*(J-1);
                     PERMUTE(MEM[S],MEM[S+1],MEM[S+2],MEM[S+3])
                  END
            END;
      IF NOT OK THEN WRITE(N#,' not understood - no action taken',13,10)
   END;   {move}
PROC GETCYCLES(XST,XFIN,SIDES);
   VAR N,PTR,ACTIVITY,J,K, LINEPOS, JJ, JK, CYLENGTH, NPOS, TAG: INTEGER;
      VC: ARRAY[288] OF INTEGER;
      START, SCOPE: ARRAY[4] OF INTEGER;
   FUNC LCM(U,V);
      FUNC GCD(M,N);
         BEGIN
            IF N=0 THEN GCD:=M
                   ELSE GCD:=GCD(N,M MOD N)
         END;   {gcd}
   BEGIN
         LCM := (U*V) DIV GCD(U,V)
   END;   {lcm}
   PROC LOOKFOR(X,Y);
      FUNC TESSNO(P);
         BEGIN
            TESSNO := (P-XST) DIV SIDES
         END;
      BEGIN
         IF M[Y]<>X THEN LOOKFOR(X,M[Y]);
         JJ := JJ + 1;
         VC[JJ] := MEM[ORG2 + M[Y]];
         FLAG[Y]:= TESSNO(Y) = TESSNO(X)
      END;
   BEGIN
      PTR:=XST;
      WHILE PTR<XFIN DO
         BEGIN
            ACTIVITY:=FALSE;
            FOR J:=1 TO SIDES DO
               BEGIN
                  K:=PTR+J-1;
```

```
                  ACTIVITY:=ACTIVITY OR ((M[K]<>K) AND FLAG[K])
               END;
            IF ACTIVITY THEN
               BEGIN
                  TAG:=0; JJ:=0;
                  FOR J:=1 TO SIDES DO
                     BEGIN
                        K:=PTR+J-1;
                        IF K<>M[K] THEN LOOKFOR(K,M[K]);
                        JJ:=JJ+1;
                        VC[JJ]:=MEM[ORG2+M[K]];
                        SCOPE[J]:=JJ-TAG;
                        TAG:=JJ
                     END;
                  START[1]:=1; CYLENGTH:=SCOPE[1];
                  FOR J:=2 TO SIDES DO
                     BEGIN
                        START[J]:=START[J-1]+SCOPE[J-1];
                        CYLENGTH:=LCM(SCOPE[J],CYLENGTH)
                     END;
                  CRLF;
                  WRITE('[');
                  LINEPOS:=1;
                  FOR J:=1 TO CYLENGTH DO
                     BEGIN
                        FOR K:=1 TO SIDES DO
                           BEGIN
                              NPOS := ((J-1) MOD SCOPE[K]) + START[K];
                              WRITE(VC[NPOS])
                           END;
                        IF J<CYLENGTH THEN
                           BEGIN
                              WRITE(',');
                              LINEPOS := LINEPOS + SIDES + 1;
                              IF LINEPOS>70 THEN
                                 BEGIN
                                    LINEPOS := 10;
                                    CRLF;
                                    WRITE('     ');
                                    WRITE('     ')
                                 END
                            END
                     END;
                  WRITE(']')
               END;
            PTR:=PTR+SIDES
         END;
      FOR N:=XST TO XFIN DO FLAG[N] := TRUE
   END;   {getcycles}
BEGIN   {main program}
   CRLF;
   DONE := FALSE;
   INITIALIZE;
```

```
   REPEAT
      RECAP := FALSE;
      OK := FALSE;
      WHILE NOT OK DO
         BEGIN
            WRITE(13,10,' Move');
            READ(FACE#);
            CRLF;
            Z(FACE)
         END;
      IF RECAP THEN
         BEGIN
            WRITE('     Move Count = ', NMOVES#,13,10);
            GETCYCLES(1,48,2);
            GETCYCLES(49,144,3);
            GETCYCLES(145,208,4)
         END
   UNTIL DONE
END.
```