Observability Analysis for Tightly-Coupled Visual-Inertial Localization and 3D Rigid-body Target Tracking

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Contents

1 Introduction

In this work, we provide an extensive observability analysis for tightly-coupled visual-inertial localization and 6DOF (position and orientation) rigid-body target tracking (VILTT). System observability is essential for state estimation, because: (1) it provides a deeper insight into a system's geometrical properties [1, 2, 3] and determines the minimal measurement modalities or state parameters needed to initialize the estimator; (2) it can be used to identify degenerate motions [4, 5, 6, 7] that cause additional unobservable directions and should be avoided or alerted whenever possible; and (3) the observability constrained (OC) estimators such as OC-EKF [1] and OC-VINS [2] that enforce the correct observability properties, can be adopted to improve consistency.

In this work, the rigid-body target is represented by a pose (orientation and position) together with feature points attached to it. The origin of the target frame is chosen from one of these target feature points as representative point, which will also be used to describe the target position. We analyze three stochastic motion models of the target, capturing most commonly-seen tracking scenarios in practice: (1) constant global linear velocity with constant local angular velocity, (2) constant local linear velocity and constant local angular velocity, and (3) a planar motion model which assumes constant local yaw rate and local planar velocity. We show that the proposed VILTT system will have at least 4 unobservable directions inherited from visual-inertial navigation system (VINS) with additional unobservable directions related to the target state based on the chosen target motion model. Geometrical interpretations for these unobservable directions are also provided and discussed.

2 Problem Formulation

2.1 Inertial State

As this work extends the standard MSCKF formulation [8], we first define the IMU state of an aided inertial navigation system (INS) as follows:

$$\mathbf{x}_I = \begin{bmatrix} {}^I_G\bar{q}^\top & \mathbf{b}_\omega^\top & {}^G\mathbf{v}_I^\top & \mathbf{b}_a^\top & {}^G\mathbf{p}_I^\top \end{bmatrix}^\top \quad (1)$$

where ${}^I_G\bar{q}$ is the unit quaternion of JPL form parameterizing the rotation ${}^I_G\mathbf{R}$ from the global frame $\{G\}$ to the current local frame $\{I\}$ [9], $\mathbf{b}_\omega$ and $\mathbf{b}_a$ are the gyroscope and accelerometer biases, and ${}^G\mathbf{v}_I$ and ${}^G\mathbf{p}_I$ are the velocity and position of the IMU expressed in the global frame, respectively. Note that the relationship between the vector quantities with true value $\mathbf{v}$, mean value $\hat{\mathbf{v}}$, and error state $\tilde{\mathbf{v}}$ takes the form $\mathbf{v} = \hat{\mathbf{v}} + \tilde{\mathbf{v}}$. For quaternions in JPL convention, with true value $\bar{q}$, mean value $\hat{\bar{q}}$, and error state $\delta\boldsymbol{\theta}$, we have $\bar{q} = [(\delta\boldsymbol{\theta}/2)^\top \ 1]^\top \otimes \hat{\bar{q}}$ with $\otimes$ as the quaternion multiplication. The error state corresponding to the INS state (1) is given as:

$$\tilde{\mathbf{x}}_I = \begin{bmatrix} {}^I\delta\boldsymbol{\theta}^\top & \tilde{\mathbf{b}}_\omega^\top & {}^G\tilde{\mathbf{v}}_I^\top & \tilde{\mathbf{b}}_a^\top & {}^G\tilde{\mathbf{p}}_I^\top \end{bmatrix}^\top \quad (2)$$

2.2 IMU Propagation

An IMU attached to the moving platform provides local linear acceleration and angular velocity measurements. In particular, the measurements $\mathbf{a}_m$ and $\boldsymbol{\omega}_m$, are related to the true values, $\mathbf{a}$ and $\boldsymbol{\omega}$, by:

$$\mathbf{a}_m = \mathbf{a} + {}^I_G\mathbf{R}\,{}^G\mathbf{g} + \mathbf{b}_a + \mathbf{n}_a \quad (3)$$

$$\boldsymbol{\omega}_m = \boldsymbol{\omega} + \mathbf{b}_\omega + \mathbf{n}_\omega \quad (4)$$

where ${}^{G}\mathbf{g} \approx [0\ 0\ 9.8]^\top$ is the true global gravity, and $\mathbf{n}_a$ and $\mathbf{n}_\omega$ are the continuous-time Gaussian noises which corrupt the measurements (note that in the rest of this paper we assume values denoted $\mathbf{n}$ to be denote zero-mean white Gaussian noises). The underlying IMU kinematics is given by [10]:

$$
\begin{aligned}
{}^{I}_{G}\dot{\bar{q}} &= \frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega}){}^{I}_{G}\bar{q} && (5)\\
{}^{G}\dot{\mathbf{v}} &= {}^{I}_{G}\mathbf{R}^\top\mathbf{a} && (6)\\
{}^{G}\dot{\mathbf{p}} &= {}^{G}\mathbf{v} && (7)\\
\dot{\mathbf{b}}_w &= \mathbf{n}_{b\omega} && (8)\\
\dot{\mathbf{b}}_a &= \mathbf{n}_{ba} && (9)
\end{aligned}
$$

$$
\boldsymbol{\Omega}(\boldsymbol{\omega}) = \begin{bmatrix} -\lfloor\boldsymbol{\omega}\rfloor & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^\top & 0 \end{bmatrix} \tag{10}
$$

where $\lfloor\cdot\rfloor$ denotes the skew symmetric matrix. Using this motion model, the standard EKF propagation step can be performed [8].

2.3 VILTT State

In our VILTT system, we have additional state parameters that we are trying to estimate concurrently with the current inertial state. We write the combined state to-be-estimate as the following:

$$
\mathbf{x} = \begin{bmatrix} \mathbf{x}_I^\top & {}^{G}\mathbf{p}_{fs}^\top & \mathbf{x}_T^\top & {}^{G}\mathbf{p}_{ft}^\top \end{bmatrix}^\top \tag{11}
$$

where $\mathbf{x}_I$, $\mathbf{x}_T$, ${}^{G}\mathbf{p}_{fs}$ and ${}^{T}\mathbf{p}_{ft}$ represents the state for the IMU, the target, the static environment feature and the target feature (rigidly attached to the target), respectively.

2.4 Target Motion Model

Each of the three target models has different state parameters that we are interested in estimating. We define $\mathbf{x}_T^{(i)}$ as the state of the i'th target model. Specifically, for target state propagation, we define the evolution of each of the three target's state as the following:

- Model 1: Given constant global linear velocity ${}^{G}\mathbf{v}_T$ and constant local angular velocity ${}^{T}\boldsymbol{\omega}$ assumption, the target state for model 1 and its evolution can be written as:

$$
\dot{\mathbf{x}}_T^{(1)} = \begin{bmatrix} {}^{T}_{G}\dot{\bar{q}} \\ {}^{T}\dot{\boldsymbol{\omega}} \\ {}^{G}\dot{\mathbf{p}}_T \\ {}^{G}\dot{\mathbf{v}}_T \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\boldsymbol{\Omega}({}^{T}\boldsymbol{\omega}){}^{T}_{G}\bar{q} \\ \mathbf{n}_{t\omega} \\ {}^{G}\mathbf{v}_T \\ \mathbf{n}_{tv} \end{bmatrix} \tag{12}
$$

- Model 2: Given the constant local linear and angular velocity ${}^{T}\mathbf{v}_T$ and ${}^{T}\boldsymbol{\omega}$, the target state for model 2 and its related evolution can be written as:

$$
\dot{\mathbf{x}}_T^{(2)} = \begin{bmatrix} {}^{T}_{G}\dot{\bar{q}} \\ {}^{T}\dot{\boldsymbol{\omega}} \\ {}^{G}\dot{\mathbf{p}}_T \\ {}^{T}\dot{\mathbf{v}}_T \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\boldsymbol{\Omega}({}^{T}\boldsymbol{\omega}){}^{T}_{G}\bar{q} \\ \mathbf{n}_{tv} \\ {}^{T}_{G}\mathbf{R}^{\top T}\mathbf{v}_T \\ \mathbf{n}_{t\omega} \end{bmatrix} \tag{13}
$$

- Model 3: Given the planar motion model with constant local planar linear velocity and constant local angular velocity around local z axis, the target state for model 3 and its evolution can be written as:

$$\dot{\mathbf{x}}_T^{(3)} = \begin{bmatrix} {}^T_G\dot{\bar{q}} \\ {}^T\dot{\omega}_z \\ {}^G\dot{\mathbf{p}}_T \\ {}^T\dot{v}_x \\ {}^T\dot{v}_y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\boldsymbol{\Omega}\left(\begin{bmatrix} n_{wx} \\ n_{wy} \\ \omega_z \end{bmatrix}\right) {}^T_G\bar{q} \\ n_{wz} \\ {}^T_G\mathbf{R}^\top \begin{bmatrix} v_x \\ v_y \\ n_{vz} \end{bmatrix} \\ n_{vx} \\ n_{vy} \end{bmatrix} \tag{14}$$

2.5 Feature Measurement - Static Environmental

As the moving IMU sensor platform moves through the environment, static environmental feature tracks are collected and used to update the state. For simplicity, we assume the camera-to-IMU extrinsic calibration is identity, so that the static feature measurement can be written as:

$$\mathbf{z}_1 = \mathbf{h}({}^G\mathbf{p}_{fs}, \mathbf{x}_I) + \mathbf{n}_{f1} \tag{15}$$

$$= \begin{bmatrix} \frac{{}^I x_{fs}}{{}^I z_{fs}} \\ \frac{{}^I y_{fs}}{{}^I z_{fs}} \end{bmatrix} + \mathbf{n}_{f1} \tag{16}$$

where ${}^I\mathbf{p}_{fs} = \begin{bmatrix} {}^I x_{fs} & {}^I y_{fs} & {}^I z_{fs} \end{bmatrix}^\top$ can be written as:

$${}^I\mathbf{p}_{fs} = {}^I_G\mathbf{R}\left({}^G\mathbf{p}_{fs} - {}^G\mathbf{p}_I\right) \tag{17}$$

2.6 Feature Measurement - Target Non-Representative

The moving IMU sensor platform also tracks features that reside on the moving target but are not of the representative point/pose of the target. We represent theses target features in the local target frame of reference and write them as the following:

$$\mathbf{z}_2 = \mathbf{h}(\mathbf{x}_I, \mathbf{x}_T, {}^T\mathbf{p}_{ft}) + \mathbf{n}_{f2} \tag{18}$$

$$= \begin{bmatrix} \frac{{}^I x_{ft}}{{}^I z_{ft}} \\ \frac{{}^I y_{ft}}{{}^I z_{ft}} \end{bmatrix} + \mathbf{n}_{f2} \tag{19}$$

where ${}^I\mathbf{p}_{ft} = \begin{bmatrix} {}^I x_{ft} & {}^I y_{ft} & {}^I z_{ft} \end{bmatrix}^\top$ can be written as:

$${}^I\mathbf{p}_{ft} = {}^I_T\mathbf{R}\left({}^T\mathbf{p}_{ft} - {}^T\mathbf{p}_I\right) \tag{20}$$

$$= {}^I_G\mathbf{R}{}^T_G\mathbf{R}^\top\left({}^T\mathbf{p}_{ft} - {}^T_G\mathbf{R}\left({}^G\mathbf{p}_I - {}^G\mathbf{p}_T\right)\right) \tag{21}$$

$$= {}^I_G\mathbf{R}\left({}^T_G\mathbf{R}^{\top T}\mathbf{p}_{ft} - \left({}^G\mathbf{p}_I - {}^G\mathbf{p}_T\right)\right) \tag{22}$$

2.7 Feature Measurement - Target Representative

A single feature is taken to be the "representative point" that the pose of the rigid-body is estimated to be at. Visual bearing measurements of this representative point are a direct measurements of the target's position, thus we can write the following:

$$\mathbf{z}_3 = \mathbf{h}(\mathbf{x}_T, \mathbf{x}_I) + \mathbf{n}_3 \tag{23}$$

$$= \begin{bmatrix} \frac{{}^I x_T}{{}^I z_T} \\ \frac{{}^I y_T}{{}^I z_T} \end{bmatrix} + \mathbf{n}_3 \tag{24}$$

Note that ${}^I\mathbf{p}_T = \begin{bmatrix} {}^I x_T & {}^I y_T & {}^I z_T \end{bmatrix}^\top$ can be described as:

$${}^I\mathbf{p}_T = {}^I_G\mathbf{R}({}^G\mathbf{p}_T - {}^G\mathbf{p}_I) \tag{25}$$

3 Observability Analysis

Following the methodology of Hesch et al. [2], we perform the observability analysis for the linearized system. The observability matrix $\mathbf{M}(\mathbf{x})$ can constructed as:

$$\mathbf{M}(\mathbf{x}) = \begin{bmatrix} \mathbf{H}_{\mathbf{x}0}\mathbf{\Phi}(0,0) \\ \mathbf{H}_{\mathbf{x}1}\mathbf{\Phi}(1,0) \\ \mathbf{H}_{\mathbf{x}2}\mathbf{\Phi}(2,0) \\ \vdots \\ \mathbf{H}_{\mathbf{x}k}\mathbf{\Phi}(k,0) \end{bmatrix} \tag{26}$$

where $\mathbf{H}_{\mathbf{x}k}$ represents the system measurement Jacobians at time step k, $\mathbf{\Phi}(k,0)$ represents the state transition matrix from time 0 to time k. The right nullspace $\mathbf{N}$ of the observability matrix $\mathbf{M}(\mathbf{x})$ spans the unobservable directions of the system. For each of the following models, we first compute the measurement Jacobian matrix $\mathbf{H}_{\mathbf{x}}$, the state transition matrix $\mathbf{\Phi}^{(i)}$, and observability matrix $\mathbf{M}^{(i)}$. From this we find the nullspace and offer their geometric interpretation. The procedure can be found in Fig. 1.

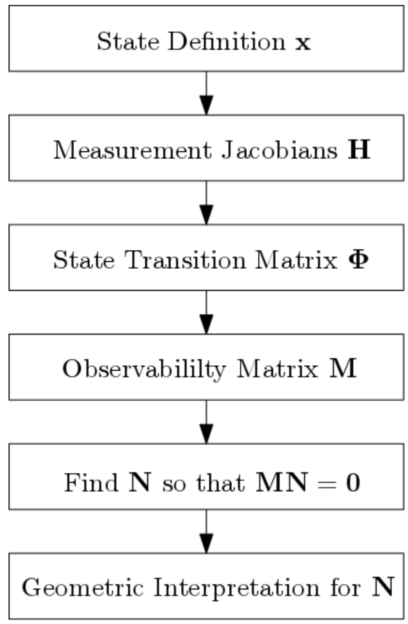

Figure 1: Visual diagram of the steps needed to perform observability analysis. For each of the target models we perform theses steps to find the unobservabile directions of the VILTT system.

4 Observability Analysis - Motion Model 1

4.1 Measurement Jacobians

According to previous sections, we have 3 measurement models: the measurement to static features in the environment $\mathbf{z}_1$, the measurement to the features in target body $\mathbf{z}_2$, and the measurement to the representative point of the rigid body $\mathbf{z}_3$. Therefore, the total measurement Jacobians can be written as:

$$\mathbf{H}_{\mathbf{x}} = \begin{bmatrix} \mathbf{H}_1 \\ \mathbf{H}_2 \\ \mathbf{H}_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}} \\ \frac{\partial \tilde{\mathbf{z}}_2}{\partial \tilde{\mathbf{x}}} \\ \frac{\partial \tilde{\mathbf{z}}_3}{\partial \tilde{\mathbf{x}}} \end{bmatrix} \tag{27}$$

where $\mathbf{H}_i$, $i = 1, 2, 3$, represent the measurement Jacobians of the 3 above measurement models. We first compute the Jacobians $\mathbf{H}_1$ as:

$$\mathbf{H}_1 = \frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}} = \begin{bmatrix} \frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}_I} & \frac{\partial \tilde{\mathbf{z}}_1}{\partial {}^G\tilde{\mathbf{p}}_{fs}} & \frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}_T^{(1)}} & \frac{\partial \tilde{\mathbf{z}}_1}{\partial {}^T\tilde{\mathbf{p}}_{ft}} \end{bmatrix} \tag{28}$$

where we have:

$$\mathbf{H}_{C1} = \frac{1}{{}^I\hat{z}_{fs}^2}\begin{bmatrix} {}^I\hat{z}_{fs} & 0 & -{}^I\hat{x}_{fs} \\ 0 & {}^I\hat{z}_{fs} & -{}^I\hat{y}_{fs} \end{bmatrix} \tag{29}$$

$$\frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}_I} = \mathbf{H}_{C1}{}^I_G\hat{\mathbf{R}}\begin{bmatrix} \lfloor {}^G\hat{\mathbf{p}}_{fs} - {}^G\hat{\mathbf{p}}_I \rfloor {}^G_I\hat{\mathbf{R}} & \mathbf{0}_{3\times 9} & -\mathbf{I}_3 \end{bmatrix} \tag{30}$$

$$\frac{\partial \tilde{\mathbf{z}}_1}{\partial {}^G\tilde{\mathbf{p}}_{fs}} = \mathbf{H}_{C1}{}^I_G\hat{\mathbf{R}}\mathbf{I}_3 \tag{31}$$

$$\frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}_T^{(1)}} = \mathbf{0}_{3\times 12} \tag{32}$$

$$\frac{\partial \tilde{\mathbf{z}}_1}{\partial {}^T\tilde{\mathbf{p}}_{fs}} = \mathbf{0}_3 \tag{33}$$

The Jacobians $\mathbf{H}_2$ can be written as:

$$\mathbf{H}_2 = \frac{\partial \tilde{\mathbf{z}}_2}{\partial \tilde{\mathbf{x}}} = \begin{bmatrix} \frac{\partial \tilde{\mathbf{z}}_2}{\partial \tilde{\mathbf{x}}_I} & \frac{\partial \tilde{\mathbf{z}}_2}{\partial {}^G\tilde{\mathbf{p}}_{ft}} & \frac{\partial \tilde{\mathbf{z}}_2}{\partial \tilde{\mathbf{x}}_T^{(1)}} & \frac{\partial \tilde{\mathbf{z}}_2}{\partial {}^T\tilde{\mathbf{p}}_{fs}} \end{bmatrix} \tag{34}$$

where we have:

$$\mathbf{H}_{C2} = \frac{1}{{}^I\hat{z}_{ft}^2}\begin{bmatrix} {}^I\hat{z}_{ft} & 0 & -{}^I\hat{x}_{ft} \\ 0 & {}^I\hat{z}_{ft} & -{}^I\hat{y}_{ft} \end{bmatrix} \tag{35}$$

$$\frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}_I} = \mathbf{H}_{C2}{}^I_G\hat{\mathbf{R}}\begin{bmatrix} \lfloor \Big({}^T_G\hat{\mathbf{R}}^\top {}^T\hat{\mathbf{p}}_{ft} - \big({}^G\hat{\mathbf{p}}_I - {}^G\hat{\mathbf{p}}_T \big) \Big) \rfloor {}^G_I\hat{\mathbf{R}} & \mathbf{0}_{3\times 9} & -\mathbf{I}_3 \end{bmatrix} \tag{36}$$

$$\frac{\partial \tilde{\mathbf{z}}_2}{\partial {}^G\tilde{\mathbf{p}}_{ft}} = \mathbf{0}_3 \tag{37}$$

$$\frac{\partial \tilde{\mathbf{z}}_2}{\partial \tilde{\mathbf{x}}_T^{(1)}} = \mathbf{H}_{C2}{}^I_G\hat{\mathbf{R}}\begin{bmatrix} -{}^T_G\hat{\mathbf{R}}^\top \lfloor {}^T\hat{\mathbf{p}}_{ft} \rfloor & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \end{bmatrix} \tag{38}$$

$$\frac{\partial \tilde{\mathbf{z}}_2}{\partial {}^T\tilde{\mathbf{p}}_{ft}} = \mathbf{H}_{C2}{}^I_G\hat{\mathbf{R}}{}^T_G\hat{\mathbf{R}}^\top \tag{39}$$

The measurement Jacobians $\mathbf{H}_3$ can be written as:

$$\mathbf{H}_3 = \frac{\partial \tilde{\mathbf{z}}_2}{\partial \tilde{\mathbf{x}}} = \begin{bmatrix} \frac{\partial \tilde{\mathbf{z}}_3}{\partial \tilde{\mathbf{x}}_I} & \frac{\partial \tilde{\mathbf{z}}_3}{\partial {}^G\tilde{\mathbf{p}}_{fs}} & \frac{\partial \tilde{\mathbf{z}}_3}{\partial \tilde{\mathbf{x}}_T^{(1)}} & \frac{\partial \tilde{\mathbf{z}}_3}{\partial {}^T\tilde{\mathbf{p}}_{ft}} \end{bmatrix} \tag{40}$$

where we have:

$$\mathbf{H}_{C3} = \frac{1}{{}^I\hat{z}_T^2}\begin{bmatrix} {}^I\hat{z}_T & 0 & -{}^I\hat{x}_T \\ 0 & {}^I\hat{z}_T & -{}^I\hat{y}_T \end{bmatrix} \tag{41}$$

$$\frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}_I} = \mathbf{H}_{C3}{}^I_G\hat{\mathbf{R}}\begin{bmatrix} \lfloor \big({}^G\hat{\mathbf{p}}_T - {}^G\hat{\mathbf{p}}_I \big) \rfloor {}^G_I\hat{\mathbf{R}} & \mathbf{0}_{3\times 9} & \mathbf{0}_3 \end{bmatrix} \tag{42}$$

$$\frac{\partial \tilde{\mathbf{z}}_3}{\partial {}^G\tilde{\mathbf{p}}_{fs}} = \mathbf{0}_3 \tag{43}$$

$$\frac{\partial \tilde{\mathbf{z}}_3}{\partial \tilde{\mathbf{x}}_T^{(1)}} = \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \end{bmatrix} \tag{44}$$

$$\frac{\partial \tilde{\mathbf{z}}_3}{\partial {}^T\tilde{\mathbf{p}}_{ft}} = \mathbf{0}_3 \tag{45}$$

4.2 State Transition Matrix

The total system state transition matrix can be written as:

$$\boldsymbol{\Phi}^{(1)} = \begin{bmatrix} \boldsymbol{\Phi}_I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Phi}_{fs} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Phi}_T^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Phi}_{ft} \end{bmatrix} \tag{46}$$

where $\boldsymbol{\Phi}_I$, $\boldsymbol{\Phi}_{fs}$, $\boldsymbol{\Phi}_T^{(1)}$ and $\boldsymbol{\Phi}_{ft}$ represent the state transition matrix for $\mathbf{x}_I$, ${}^G\mathbf{p}_{fs}$, $\mathbf{x}_T^{(1)}$ and ${}^T\mathbf{p}_{ft}$, respectively. Note that $\boldsymbol{\Phi}_{fs} = \mathbf{I}_3$, $\boldsymbol{\Phi}_{ft} = \mathbf{I}_3$ and $\boldsymbol{\Phi}_I$ can be written from [2] as:

$$\boldsymbol{\Phi}_I(k,0) = \begin{bmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \boldsymbol{\Phi}_{31} & \boldsymbol{\Phi}_{32} & \mathbf{I}_3 & \boldsymbol{\Phi}_{34} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \boldsymbol{\Phi}_{51} & \boldsymbol{\Phi}_{52} & \boldsymbol{\Phi}_{53} & \boldsymbol{\Phi}_{54} & \mathbf{I}_3 \end{bmatrix} \tag{47}$$

where the related items in the matrix can be written as:

$$\boldsymbol{\Phi}_{I11} = {}^{I_k}_{I_0}\mathbf{R} \tag{48}$$

$$\boldsymbol{\Phi}_{I31} = -\lfloor \left({}^G\mathbf{v}_{I_k} - {}^G\mathbf{v}_{I_0}\right) + {}^G\mathbf{g}\delta t_k \times \rfloor {}^G_{I_0}\mathbf{R} \tag{49}$$

$$\boldsymbol{\Phi}_{I51} = \lfloor {}^G\mathbf{p}_{I_0} + {}^G\mathbf{v}_{I_0}\delta t_k - \frac{1}{2}{}^G\mathbf{g}\delta t_k^2 - {}^G\mathbf{p}_{I_k} \times \rfloor {}^G_{I_0}\mathbf{R} \tag{50}$$

$$\boldsymbol{\Phi}_{I12} = -\int_{t_0}^{t_k} {}^{I_\tau}_{I_k}\mathbf{R}^\top \mathbf{d}\tau \tag{51}$$

$$\boldsymbol{\Phi}_{I32} = \int_{t_0}^{t_k} {}^{I_s}_{G}\mathbf{R}^\top \lfloor {}^{I_s}\mathbf{a}\times \rfloor \int_{t_0}^{s} {}^{I_\tau}_{I_s}\mathbf{R}^\top \mathbf{d}\tau \mathbf{d}s \tag{52}$$

$$\boldsymbol{\Phi}_{I52} = \int_{t_0}^{t_k}\int_{t_0}^{\theta} {}^{I_s}_{G}\mathbf{R}^\top \lfloor {}^{I_s}\mathbf{a}\times \rfloor \int_{t_0}^{t_s} {}^{I_\tau}_{I_s}\mathbf{R}^\top \mathbf{d}\tau \mathbf{d}s\mathbf{d}\theta \tag{53}$$

$$\boldsymbol{\Phi}_{I53} = \mathbf{I}_3\delta t_k \tag{54}$$

$$\boldsymbol{\Phi}_{I34} = -\int_{t_0}^{t_k} {}^{I_\tau}_{G}\mathbf{R}^\top \mathbf{d}\tau \tag{55}$$

$$\boldsymbol{\Phi}_{I54} = -\int_{t_0}^{t_k}\int_{t_0}^{t_s} {}^{I_\tau}_{G}\mathbf{R}^\top \mathbf{d}\tau\mathbf{d}s \tag{56}$$

Now we need to solve for the remaining target state transition matrix $\boldsymbol{\Phi}_T^{(1)}$. The linearized system for the target motion model (12) can be written as:

$$\dot{\tilde{x}}_T^{(1)} = \begin{bmatrix} \delta\dot{\boldsymbol{\theta}}_T \\ {}^T\dot{\tilde{\boldsymbol{\omega}}} \\ {}^G\dot{\tilde{\mathbf{p}}}_T \\ {}^G\dot{\tilde{\mathbf{v}}} \end{bmatrix} \simeq \begin{bmatrix} -\lfloor {}^T\hat{\boldsymbol{\omega}} \rfloor & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{\theta}_T \\ {}^T\tilde{\boldsymbol{\omega}} \\ {}^G\tilde{\mathbf{p}}_T \\ {}^G\tilde{\mathbf{v}} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{n}_{t\omega} \\ \mathbf{n}_{tv} \end{bmatrix} \tag{57}$$

Thus, the state transition $\boldsymbol{\Phi}_T^{(1)}$ evolution can be written as:

$$\dot{\boldsymbol{\Phi}}_T^{(1)} = \begin{bmatrix} -\lfloor {}^T\hat{\boldsymbol{\omega}} \rfloor & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi}_{T11} & \boldsymbol{\Phi}_{T12} & \boldsymbol{\Phi}_{T13} & \boldsymbol{\Phi}_{T14} \\ \boldsymbol{\Phi}_{T21} & \boldsymbol{\Phi}_{T22} & \boldsymbol{\Phi}_{T23} & \boldsymbol{\Phi}_{T24} \\ \boldsymbol{\Phi}_{T31} & \boldsymbol{\Phi}_{T32} & \boldsymbol{\Phi}_{T33} & \boldsymbol{\Phi}_{T34} \\ \boldsymbol{\Phi}_{T41} & \boldsymbol{\Phi}_{T42} & \boldsymbol{\Phi}_{T43} & \boldsymbol{\Phi}_{T44} \end{bmatrix} \tag{58}$$

Then, the target state transition matrix can be solved as:

$$\mathbf{\Phi}_T^{(1)} = \begin{bmatrix} \mathbf{\Phi}_{T11} & \mathbf{\Phi}_{T12} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{\Phi}_{T34} \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{59}$$

where we have:

$$\mathbf{\Phi}_{T11} = {}^{T_k}_{T_0}\mathbf{R} \tag{60}$$

$$\mathbf{\Phi}_{T12} = \int_{t_0}^{t} {}^{T_k}_{T_\tau}\mathbf{R}d\tau \tag{61}$$

$$\mathbf{\Phi}_{T34} = \mathbf{I}_3\delta t_k \tag{62}$$

Therefore, the target state transition matrix can be written as:

$$\mathbf{\Phi}_T^{(1)} = \begin{bmatrix} {}^{T_k}_{T_0}\mathbf{R} & \int_{t_0}^{t} {}^{T_k}_{T_\tau}\mathbf{R}d\tau & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{I}_3\delta t_k \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{63}$$

4.3 Observability Matrix

With abuse of notation, the k'th block of the observability matrix can be written as:

$$\mathbf{M}_k^{(1)} = \mathbf{H}_{\mathbf{x}k}\mathbf{\Phi}^{(1)}(k,0) = \mathbf{H}_{\mathbf{x}k} \begin{bmatrix} \mathbf{\Phi}_I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Phi}_{fs} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Phi}_T^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{\Phi}_{ft} \end{bmatrix} \tag{64}$$

$$= \begin{bmatrix} \mathbf{H}_{C1}{}^I_G\hat{\mathbf{R}} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{H}_{C2}{}^I_G\hat{\mathbf{R}} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}_{C3}{}^I_G\hat{\mathbf{R}} \end{bmatrix} \times$$

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} & \mathbf{M}_{14} & \mathbf{M}_{15} & \mathbf{M}_{16} & \mathbf{M}_{17} & \mathbf{M}_{18} & \mathbf{M}_{19} & \mathbf{M}_{1,10} & \mathbf{M}_{1,11} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} & \mathbf{M}_{24} & \mathbf{M}_{25} & \mathbf{M}_{26} & \mathbf{M}_{27} & \mathbf{M}_{28} & \mathbf{M}_{29} & \mathbf{M}_{2,10} & \mathbf{M}_{2,11} \end{bmatrix} \tag{65}$$

The first row of this matrix can be denoted as:

$$\mathbf{M}_{11} = \lfloor {}^G\hat{\mathbf{p}}_{fs} - {}^G\hat{\mathbf{p}}_{I_k} \rfloor {}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Phi}_{I11} - \mathbf{\Phi}_{I51} = \lfloor {}^G\hat{\mathbf{p}}_{fs} - {}^G\hat{\mathbf{p}}_{I_0} - {}^G\hat{\mathbf{v}}_{I_0}\delta t_k + \frac{1}{2}{}^G\mathbf{g}\delta t_k^2 \rfloor {}^G_{I_0}\hat{\mathbf{R}} \tag{66}$$

$$\mathbf{M}_{12} = \lfloor {}^G\hat{\mathbf{p}}_{fs} - {}^G\hat{\mathbf{p}}_{I_k} \rfloor {}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Phi}_{I12} - \mathbf{\Phi}_{I52} \tag{67}$$

$$\mathbf{M}_{13} = -\mathbf{\Phi}_{I53} = -\mathbf{I}_3\delta t_k \tag{68}$$

$$\mathbf{M}_{14} = -\mathbf{\Phi}_{I54} \tag{69}$$

$$\mathbf{M}_{15} = -\mathbf{I}_3 \tag{70}$$

$$\mathbf{M}_{16} = \mathbf{I}_3 \tag{71}$$

$$\mathbf{M}_{17} = \mathbf{M}_{18} = \mathbf{M}_{19} = \mathbf{M}_{1,10} = \mathbf{M}_{1,11} = \mathbf{0}_3 \tag{72}$$

The second row of this matrix can be described as:

$$\mathbf{M}_{21} = \lfloor {}^{G}_{T_k}\hat{\mathbf{R}}^T \hat{\mathbf{p}}_{ft} - {}^{G}\hat{\mathbf{p}}_{I_k} + {}^{G}\hat{\mathbf{p}}_{T_k} \rfloor {}^{G}_{I_k}\hat{\mathbf{R}} \mathbf{\Phi}_{I11} - \mathbf{\Phi}_{I51} \tag{73}$$

$$= \lfloor {}^{G}_{T_k}\hat{\mathbf{R}}^T \hat{\mathbf{p}}_{\mathbf{f}} + {}^{G}\hat{\mathbf{p}}_{T_k} - {}^{G}\hat{\mathbf{p}}_{I_0} - {}^{G}\hat{\mathbf{v}}_{I_0}\delta t_k + \frac{1}{2}{}^{G}\mathbf{g}\delta t_k^2 \rfloor {}^{G}_{I_0}\hat{\mathbf{R}} \tag{74}$$

$$\mathbf{M}_{22} = \lfloor {}^{G}_{T_k}\hat{\mathbf{R}}^T \hat{\mathbf{p}}_{ft} - {}^{G}\hat{\mathbf{p}}_{I_k} + {}^{G}\hat{\mathbf{p}}_{T_k} \rfloor {}^{G}_{I}\hat{\mathbf{R}} \mathbf{\Phi}_{I12} - \mathbf{\Phi}_{I52} \tag{75}$$

$$\mathbf{M}_{23} = -\mathbf{\Phi}_{I53} = -\mathbf{I}_3\delta t_k \tag{76}$$

$$\mathbf{M}_{24} = -\mathbf{\Phi}_{I54} \tag{77}$$

$$\mathbf{M}_{25} = -\mathbf{I}_3 \tag{78}$$

$$\mathbf{M}_{26} = \mathbf{0}_3 \tag{79}$$

$$\mathbf{M}_{27} = -{}^{G}_{T_k}\hat{\mathbf{R}} \lfloor {}^{T}\hat{\mathbf{p}}_{ft} \rfloor \mathbf{\Phi}_{T11} = \lfloor -{}^{G}_{T_k}\mathbf{R}^T \mathbf{p}_{ft} \rfloor {}^{G}_{T_0}\hat{\mathbf{R}} \tag{80}$$

$$\mathbf{M}_{28} = -{}^{G}_{T_k}\hat{\mathbf{R}} \lfloor {}^{T}\hat{\mathbf{p}}_{ft} \rfloor \mathbf{\Phi}_{T12} \tag{81}$$

$$\mathbf{M}_{29} = \mathbf{I}_3 \tag{82}$$

$$\mathbf{M}_{2,10} = \mathbf{\Phi}_{T34} = \mathbf{I}_3\delta t_k \tag{83}$$

$$\mathbf{M}_{2,11} = {}^{G}_{T_k}\hat{\mathbf{R}} \tag{84}$$

The third row of this matrix can be described as:

$$\mathbf{M}_{31} = \lfloor -{}^{G}\hat{\mathbf{p}}_{I_k} + {}^{G}\hat{\mathbf{p}}_{T_k} \rfloor {}^{G}_{I_k}\hat{\mathbf{R}} \mathbf{\Phi}_{I11} - \mathbf{\Phi}_{I51} = \lfloor {}^{G}\hat{\mathbf{p}}_{T_k} - {}^{G}\hat{\mathbf{p}}_{I_0} - {}^{G}\hat{\mathbf{v}}_{I_0}\delta t_k + \frac{1}{2}{}^{G}\mathbf{g}\delta t_k^2 \rfloor {}^{G}_{I_0}\mathbf{R} \tag{85}$$

$$\mathbf{M}_{32} = \lfloor -{}^{G}\hat{\mathbf{p}}_{I_k} + {}^{G}\hat{\mathbf{p}}_{T_k} \rfloor {}^{G}_{I}\mathbf{R} \mathbf{\Phi}_{I12} - \mathbf{\Phi}_{I52} \tag{86}$$

$$\mathbf{M}_{33} = -\mathbf{\Phi}_{I53} = -\mathbf{I}_3\delta t_k \tag{87}$$

$$\mathbf{M}_{34} = -\mathbf{\Phi}_{I54} \tag{88}$$

$$\mathbf{M}_{35} = -\mathbf{I}_3 \tag{89}$$

$$\mathbf{M}_{36} = \mathbf{0}_3 \tag{90}$$

$$\mathbf{M}_{37} = \mathbf{0}_3 \tag{91}$$

$$\mathbf{M}_{38} = \mathbf{0} \tag{92}$$

$$\mathbf{M}_{39} = \mathbf{I}_3 \tag{93}$$

$$\mathbf{M}_{3,10} = \mathbf{\Phi}_{T34} = \mathbf{I}_3\delta t_k \tag{94}$$

$$\mathbf{M}_{3,11} = \mathbf{0}_3 \tag{95}$$

Therefore, we can rewrite the k's block of the observability matrix as:

$$\mathbf{M}_k^{(1)} = \mathbf{H}_{\mathbf{x}k} \mathbf{\Phi}^{(1)}(k, 0) \tag{96}$$

$$= \begin{bmatrix} \mathbf{H}_{C1}{}^{I}_{G}\hat{\mathbf{R}} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{H}_{C2}{}^{I}_{G}\hat{\mathbf{R}} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}_{C3}{}^{I}_{G}\hat{\mathbf{R}} \end{bmatrix} \times$$

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & -\mathbf{I}_3\delta t_k & \mathbf{M}_{14} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{M}_{21} & \mathbf{M}_{22} & -\mathbf{I}_3\delta t_k & \mathbf{M}_{24} & -\mathbf{I}_3 & \mathbf{0}_3 & \lfloor -{}^{G}_{T_k}\hat{\mathbf{R}}^T \hat{\mathbf{p}}_{\mathbf{f}} \rfloor {}^{G}_{T_0}\hat{\mathbf{R}} & \mathbf{M}_{28} & \mathbf{I}_3 & \mathbf{I}_3\delta t_k & {}^{G}_{T_k}\hat{\mathbf{R}} \\ \mathbf{M}_{31} & \mathbf{M}_{32} & -\mathbf{I}_3\delta t_k & \mathbf{M}_{24} & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{I}_3\delta t_k & \mathbf{0}_3 \end{bmatrix} \tag{97}$$

Note that given the assumption that constant global velocity ${}^{G}\hat{\mathbf{v}}_T$, we consider the ideal noise free case in observability analysis, the ${}^{G}\hat{\mathbf{p}}_{T_k}$ can be written as:

$${}^{G}\hat{\mathbf{p}}_{T_k} = {}^{G}\hat{\mathbf{p}}_{T_0} + {}^{G}\hat{\mathbf{v}}_T\delta t_k \tag{98}$$

Based on the constant ${}^{T}\hat{\boldsymbol{\omega}}$ assumption, we can have that:

$$\boldsymbol{\Phi}_{T12}\lfloor {}^{T}\hat{\boldsymbol{\omega}}\rfloor = \int_{t_0}^{t} {}^{T_k}_{T_\tau}\hat{\mathbf{R}}\mathbf{d}\tau\lfloor {}^{T}\hat{\boldsymbol{\omega}}\rfloor = \mathbf{I}_3 - {}^{T_k}_{T_0}\hat{\mathbf{R}} \tag{99}$$

$${}^{G}_{T_k}\hat{\mathbf{R}}^{T}\hat{\boldsymbol{\omega}} = {}^{G}_{T_0}\hat{\mathbf{R}}{}^{T_0}_{T_k}\hat{\mathbf{R}}^{T}\boldsymbol{\omega} = {}^{G}_{T_0}\hat{\mathbf{R}}\left(\mathbf{I}_3 + \frac{\sin|\boldsymbol{\theta}(\delta t_k)|}{|{}^{T}\hat{\boldsymbol{\omega}}|}\lfloor {}^{T}\hat{\boldsymbol{\omega}}\rfloor + \frac{1-\cos|\boldsymbol{\theta}(\delta t_k)|}{|{}^{T}\hat{\boldsymbol{\omega}}|^2}\lfloor {}^{T}\hat{\boldsymbol{\omega}}\rfloor^2\right){}^{T}\hat{\boldsymbol{\omega}} = {}^{G}_{T_0}\hat{\mathbf{R}}^{T}\hat{\boldsymbol{\omega}} \tag{100}$$

The unobservable directions $\mathbf{N}^{(1)}$ span the right null space of the observability matrix $\mathbf{M}^{(1)}$, that is: $\mathbf{M}^{(1)}\mathbf{N}^{(1)} = \mathbf{0}$. Based on the derivation of the observability matrix, we can have the unobservable directions as:

$$\mathbf{N}^{(1)} = \begin{bmatrix}\mathbf{N}_1^{(1)} & \mathbf{N}_{2:4}^{(1)} & \mathbf{N}_{{}^{T}_{G}\mathbf{R}}^{(1)}\end{bmatrix} = \begin{bmatrix} {}^{I_0}_{G}\hat{\mathbf{R}}^{G}\mathbf{g} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^{G}\hat{\mathbf{v}}_{I_0}\rfloor^{G}\mathbf{g} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^{G}\hat{\mathbf{p}}_{I_0}\rfloor^{G}\mathbf{g} & \mathbf{I}_3 & \mathbf{0}_3 \\ -\lfloor {}^{G}\hat{\mathbf{p}}_{fs}\rfloor^{G}\mathbf{g} & \mathbf{I}_3 & \mathbf{0}_3 \\ -{}^{T_0}_{I_0}\hat{\mathbf{R}}^{G}\mathbf{g} & \mathbf{0}_3 & \mathbf{I}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \lfloor {}^{T}\hat{\boldsymbol{\omega}}\rfloor \\ -\lfloor {}^{G}\hat{\mathbf{p}}_{T_0}\rfloor^{G}\mathbf{g} & \mathbf{I}_3 & \mathbf{0}_3 \\ -\lfloor {}^{G}\hat{\mathbf{v}}_{T_0}\rfloor^{G}\mathbf{g} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \lfloor {}^{T}\hat{\mathbf{p}}_{ft}\rfloor \end{bmatrix} \tag{101}$$

Note that the $\mathbf{N}_{1:4}^{(1)}$ relate to the global yaw and the global IMU position. $\mathbf{N}_{{}^{T}_{G}\mathbf{R}}^{(1)}$ relate to target body orientation.

But, if without the direct target representative point measurement (due to occlusion), we will have one additional unobservable direction related the representative point position as:

$$\mathbf{N}_{{}^{G}\mathbf{p}_T}^{(1)} = \begin{bmatrix}\mathbf{0}_{1\times 15} & \mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 3} & \left({}^{G}_{T_0}\hat{\mathbf{R}}^{T}\hat{\boldsymbol{\omega}}\right)^{\top} & \mathbf{0}_{1\times 3} & -\left({}^{T}\hat{\boldsymbol{\omega}}\right)^{\top}\end{bmatrix}^{\top} \tag{102}$$

4.4 Geometrical Interpretation - Verification for $\mathbf{N}_{{}^{T}_{G}\mathbf{R}}$

If we disturb the target orientation by a small angle vector $\delta\boldsymbol{\phi}$, then we will have the disturbed target state parameters as:

$${}^{T'}_{G}\mathbf{R} = {}^{T'}_{T}\mathbf{R}{}^{T}_{G}\mathbf{R} \simeq (\mathbf{I}_3 - \lfloor\delta\boldsymbol{\phi}\rfloor)\,{}^{T}_{G}\mathbf{R} \tag{103}$$

$${}^{T'}\omega = {}^{T'}_{T}\mathbf{R}^{T}\omega \simeq (\mathbf{I}_3 - \lfloor\delta\boldsymbol{\phi}\rfloor)\,{}^{T}\boldsymbol{\omega} = {}^{T}\boldsymbol{\omega} + \lfloor {}^{T}\boldsymbol{\omega}\rfloor\delta\boldsymbol{\phi} \tag{104}$$

$${}^{G}\mathbf{p}_{T'} = {}^{G}\mathbf{p}_T \tag{105}$$

$${}^{G}\mathbf{v}_{T'} = {}^{G}\mathbf{v}_T \tag{106}$$

$${}^{T'}\mathbf{p}_{ft} = {}^{T'}_{T}\mathbf{R}^{T}\mathbf{p}_{ft} \simeq (\mathbf{I}_3 - \lfloor\delta\boldsymbol{\phi}\rfloor)\,{}^{T}\mathbf{p}_{ft} = {}^{T}\mathbf{p}_{ft} + \lfloor {}^{T}\mathbf{p}_{ft}\rfloor\delta\boldsymbol{\phi} \tag{107}$$

Then, we can have:

$${}^{I}\mathbf{p}_{ft'} = {}^{I}_{G}\mathbf{R}\left({}^{G}_{T'}\mathbf{R}^{T'}\mathbf{p}_{ft} - {}^{G}\mathbf{p}_I + {}^{G}\mathbf{p}_{T'}\right) = {}^{I}_{G}\mathbf{R}\left({}^{G}_{T}\mathbf{R}{}^{T}_{T'}\mathbf{R}{}^{T'}_{T}\mathbf{R}^{T}\mathbf{p}_{ft} - {}^{G}\mathbf{p}_I + {}^{G}\mathbf{p}_T\right) \tag{108}$$

$$= {}^{I}\mathbf{p}_{ft} \tag{109}$$

$${}^{I}\mathbf{p}_{T'} = {}^{I}_{G}\mathbf{R}\left({}^{G}\mathbf{p}_{T'} - {}^{G}\mathbf{p}_I\right) = {}^{I}_{G}\mathbf{R}\left({}^{G}\mathbf{p}_T - {}^{G}\mathbf{p}_I\right) \tag{110}$$

$$= {}^{I}\mathbf{p}_T \tag{111}$$

Based on (15)(18)(23), even with the orientation disturbance $\delta\boldsymbol{\phi}$, the system will still have the same target feature and target representative point measurements. Therefore, even with measurements, the system still cannot distinguish the ambiguity caused by this disturbance.

Since we assume $\delta\boldsymbol{\phi}$ is a small perturbation, we can linearize the disturbed system state vector $\mathbf{x}'$ at current state estimate $\hat{\mathbf{x}}$. Thus, the related error states can be written as:

$$\tilde{\mathbf{x}}' = \begin{bmatrix} \tilde{\mathbf{x}}'_I \\ {}^G\tilde{\mathbf{p}}'_{fs} \\ \delta\theta_{T'} \\ {}^{T'}\tilde{\boldsymbol{\omega}} \\ {}^G\tilde{\mathbf{p}}_{T'} \\ {}^G\tilde{\mathbf{v}}_{T'} \\ {}^{T'}\tilde{\mathbf{p}}_{ft} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{x}}_I \\ {}^G\tilde{\mathbf{p}}_{fs} \\ \delta\theta_T + \delta\boldsymbol{\phi} \\ {}^T\tilde{\boldsymbol{\omega}} + \lfloor {}^T\hat{\boldsymbol{\omega}} \rfloor \delta\boldsymbol{\phi} \\ {}^G\tilde{\mathbf{p}}_T \\ {}^G\tilde{\mathbf{v}}_T \\ {}^T\tilde{\mathbf{p}}_{ft} + \lfloor {}^T\hat{\mathbf{p}}_{ft} \rfloor \delta\boldsymbol{\phi} \end{bmatrix} = \tilde{\mathbf{x}} + \begin{bmatrix} \mathbf{0}_{18\times 3} \\ \mathbf{I}_3 \\ \lfloor {}^T\hat{\boldsymbol{\omega}} \rfloor \\ \mathbf{0}_3 \\ \mathbf{0}_3 \\ \lfloor {}^T\hat{\mathbf{p}}_{ft} \rfloor \end{bmatrix} \delta\boldsymbol{\phi} = \tilde{\mathbf{x}} + \mathbf{N}^{(1)}_{{}^T_G\mathbf{R}} \delta\boldsymbol{\phi} \tag{112}$$

It can be seen that the disturbance exactly follow the unobservable directions related to the target orientation.

4.5 Geometrical Interpretation - Verification for $\mathbf{N}^{(1)}_{{}^G\mathbf{p}_T}$

Next, we verify $\mathbf{N}^{(1)}_{{}^G\mathbf{p}_T}$. Without the representative point measurement $\mathbf{z}_3$, if we disturb the target representative point position with δp along the direction of the rotation axis ${}^G_{T_0}\mathbf{R}^T\boldsymbol{\omega}$, then we can have the disturbed state vector as:

$$ {}^{T'}_G\mathbf{R} = {}^T_G\mathbf{R} \tag{113}$$

$$ {}^{T'}\boldsymbol{\omega} = {}^T\boldsymbol{\omega} \tag{114}$$

$$ {}^G\mathbf{p}_{T'} = {}^G\mathbf{p}_T + {}^G_{T_0}\mathbf{R}^T\boldsymbol{\omega}\delta p \tag{115}$$

$$ \Rightarrow {}^T\mathbf{p}_{T'} = {}^T_G\mathbf{R}\left({}^G\mathbf{p}_{T'} - {}^G\mathbf{p}_T\right) = {}^T_{T_0}\mathbf{R}^T\boldsymbol{\omega}\delta p = {}^T\boldsymbol{\omega}\delta p \tag{116}$$

$$ {}^G\mathbf{v}_{T'} = {}^G\mathbf{v}_T \tag{117}$$

$$ {}^{T'}\mathbf{p}_{ft} = {}^{T'}_T\mathbf{R}\left({}^T\mathbf{p}_{ft} - {}^T\mathbf{p}_{T'}\right) = {}^T\mathbf{p}_{ft} - {}^T\boldsymbol{\omega}\delta p \tag{118}$$

If we only have target feature measurements, then we have:

$$ {}^I\mathbf{p}_{ft'} = {}^I_G\mathbf{R}\left({}^G_{T'}\mathbf{R}^{T'}\mathbf{p}_{ft} - {}^G\mathbf{p}_I + {}^G\mathbf{p}_{T'}\right) \tag{119}$$

$$ = {}^I_G\mathbf{R}\left({}^G_T\mathbf{R}^T_{T'}\mathbf{R}^{T'}_T\mathbf{R}\left({}^T\mathbf{p}_{ft} - {}^T\boldsymbol{\omega}\delta p\right) - {}^G\mathbf{p}_I + {}^G\mathbf{p}_T + {}^G_{T_0}\mathbf{R}^T\boldsymbol{\omega}\delta p\right) \tag{120}$$

$$ = {}^I\mathbf{p}_{ft} \tag{121}$$

Based on (15)(18), the system will still have the same target feature measurements. That means the ambiguity caused by the target position disturbance cannot be distinguished by the measurements. Since δp is assumed to be a small perturbation, if we linearized the disturbed system state $\mathbf{x}'$ at current state estimate $\hat{\mathbf{x}}$, then the related error states can be written as:

$$\tilde{\mathbf{x}}' = \begin{bmatrix} \tilde{\mathbf{x}}'_I \\ {}^G\tilde{\mathbf{p}}'_{fs} \\ \delta\theta_{T'} \\ {}^{T'}\tilde{\boldsymbol{\omega}} \\ {}^G\tilde{\mathbf{p}}_{T'} \\ {}^G\tilde{\mathbf{v}}_{T'} \\ {}^{T'}\tilde{\mathbf{p}}_{ft} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{x}}_I \\ {}^G\tilde{\mathbf{p}}_{fs} \\ \delta\theta_T \\ {}^T\tilde{\boldsymbol{\omega}} \\ {}^G\tilde{\mathbf{p}}_T + {}^G_{T_0}\hat{\mathbf{R}}^T\boldsymbol{\omega}\delta p \\ {}^G\tilde{\mathbf{v}}_T \\ {}^T\tilde{\mathbf{p}}_{ft} - {}^T\hat{\boldsymbol{\omega}}\delta p \end{bmatrix} = \tilde{\mathbf{x}} + \begin{bmatrix} \mathbf{0}_{18\times 3} \\ \mathbf{0}_3 \\ \mathbf{0}_3 \\ {}^G_{T_0}\hat{\mathbf{R}}^T\boldsymbol{\omega} \\ \mathbf{0}_3 \\ -{}^T\hat{\boldsymbol{\omega}} \end{bmatrix} \delta p = \tilde{\mathbf{x}} + \mathbf{N}^{(1)}_{{}^G\mathbf{p}_T}\delta p \tag{122}$$

Similarly, it can be seen that the disturbance exactly follow the unobservable directions related to the target representative point position.

5 Observability Analysis - Motion Model 2

5.1 Measurement Jacobians

With abuse of notation, the total system measurements for motion model 2 can still be written as:

$$\mathbf{H_x} = \begin{bmatrix} \mathbf{H}_1 \\ \mathbf{H}_2 \\ \mathbf{H}_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}} \\ \frac{\partial \tilde{\mathbf{z}}_2}{\partial \tilde{\mathbf{x}}} \\ \frac{\partial \tilde{\mathbf{z}}_3}{\partial \tilde{\mathbf{x}}} \end{bmatrix} \tag{123}$$

Based on the motion model 2 for target rigid body, the measurement Jacobians for the static environment feature $\mathbf{H}_1$, the target feature $\mathbf{H}_2$ and the target representative point measurements $\mathbf{H}_3$ can be computed respectively. The Jacobians of the measurement to the environmental feature $\mathbf{H}_1$ can be written as:

$$\mathbf{H}_1 = \frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}} = \begin{bmatrix} \frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}_I} & \frac{\partial \tilde{\mathbf{z}}_1}{\partial {}^G\tilde{\mathbf{p}}_{fs}} & \frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}_T^{(2)}} & \frac{\partial \tilde{\mathbf{z}}_1}{\partial {}^T\tilde{\mathbf{p}}_{ft}} \end{bmatrix} \tag{124}$$

where we have:

$$\mathbf{H}_{C1} = \frac{1}{{}^I\hat{z}_{fs}^2} \begin{bmatrix} {}^I\hat{z}_{fs} & 0 & -{}^I\hat{x}_{fs} \\ 0 & {}^I\hat{z}_{fs} & -{}^I\hat{y}_{fs} \end{bmatrix} \tag{125}$$

$$\frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}_I} = \mathbf{H}_{C1} {}^I_G\hat{\mathbf{R}} \begin{bmatrix} \lfloor {}^G\hat{\mathbf{p}}_{fs} - {}^G\hat{\mathbf{p}}_I \rfloor {}^G_I\hat{\mathbf{R}} & \mathbf{0}_{3\times 9} & -\mathbf{I}_3 \end{bmatrix} \tag{126}$$

$$\frac{\partial \tilde{\mathbf{z}}_1}{\partial {}^G\tilde{\mathbf{p}}_{fs}} = \mathbf{H}_{C1} {}^I_G\hat{\mathbf{R}} \mathbf{I}_3 \tag{127}$$

$$\frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}_T^{(2)}} = \mathbf{0}_{3\times 12} \tag{128}$$

$$\frac{\partial \tilde{\mathbf{z}}_1}{\partial {}^T\tilde{\mathbf{p}}_{ft}} = \mathbf{0}_3 \tag{129}$$

The target feature measurement Jacobians $\mathbf{H}_2$ can be written as:

$$\mathbf{H}_2 = \frac{\partial \tilde{\mathbf{z}}_2}{\partial \tilde{\mathbf{x}}} = \begin{bmatrix} \frac{\partial \tilde{\mathbf{z}}_2}{\partial \tilde{\mathbf{x}}_I} & \frac{\partial \tilde{\mathbf{z}}_2}{\partial {}^G\tilde{\mathbf{p}}_{fs}} & \frac{\partial \tilde{\mathbf{z}}_2}{\partial \tilde{\mathbf{x}}_T^{(2)}} & \frac{\partial \tilde{\mathbf{z}}_2}{\partial {}^T\tilde{\mathbf{p}}_{ft}} \end{bmatrix} \tag{130}$$

where we have:

$$\mathbf{H}_{C2} = \frac{1}{{}^I\hat{z}_{ft}^2} \begin{bmatrix} {}^I\hat{z}_{ft} & 0 & -{}^I\hat{x}_{ft} \\ 0 & {}^I\hat{z}_{ft} & -{}^I\hat{y}_{ft} \end{bmatrix} \tag{131}$$

$$\frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}_I} = \mathbf{H}_{C2} {}^I_G\hat{\mathbf{R}} \begin{bmatrix} \lfloor \left({}^T_G\hat{\mathbf{R}}^{\top T}\hat{\mathbf{p}}_{ft} - \left({}^G\hat{\mathbf{p}}_I - {}^G\hat{\mathbf{p}}_T \right) \right) \rfloor {}^G_I\hat{\mathbf{R}} & \mathbf{0}_{3\times 9} & -\mathbf{I}_3 \end{bmatrix} \tag{132}$$

$$\frac{\partial \tilde{\mathbf{z}}_2}{\partial {}^G\tilde{\mathbf{p}}_{ft}} = \mathbf{0}_3 \tag{133}$$

$$\frac{\partial \tilde{\mathbf{z}}_2}{\partial \tilde{\mathbf{x}}_T^{(2)}} = \mathbf{H}_{C2} {}^I_G\hat{\mathbf{R}} \begin{bmatrix} -{}^T_G\hat{\mathbf{R}}^\top \lfloor {}^T\hat{\mathbf{p}}_{ft} \rfloor & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \end{bmatrix} \tag{134}$$

$$\frac{\partial \tilde{\mathbf{z}}_2}{\partial {}^T\tilde{\mathbf{p}}_{ft}} = \mathbf{H}_{C2} {}^I_G\hat{\mathbf{R}} {}^T_G\hat{\mathbf{R}}^\top \tag{135}$$

The target representative point measurement Jacobians $\mathbf{H}_3$ can be written as:

$$\mathbf{H}_3 = \frac{\partial \tilde{\mathbf{z}}_2}{\partial \tilde{\mathbf{x}}} = \begin{bmatrix} \frac{\partial \tilde{\mathbf{z}}_3}{\partial \tilde{\mathbf{x}}_I} & \frac{\partial \tilde{\mathbf{z}}_3}{\partial {}^G\tilde{\mathbf{p}}_{fs}} & \frac{\partial \tilde{\mathbf{z}}_3}{\partial \tilde{\mathbf{x}}_T^{(2)}} & \frac{\partial \tilde{\mathbf{z}}_3}{\partial {}^T\tilde{\mathbf{p}}_{ft}} \end{bmatrix} \tag{136}$$

where we have:

$$\mathbf{H}_{C3} = \frac{1}{{}^I\hat{z}_T^2} \begin{bmatrix} {}^I\hat{z}_T & 0 & -{}^I\hat{x}_T \\ 0 & {}^I\hat{z}_T & -{}^I\hat{y}_T \end{bmatrix} \tag{137}$$

$$\frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}_I} = \mathbf{H}_{C3} {}^I_G\hat{\mathbf{R}} \begin{bmatrix} \lfloor \left({}^G\hat{\mathbf{p}}_T - {}^G\hat{\mathbf{p}}_I \right) \rfloor {}^G_I\hat{\mathbf{R}} & \mathbf{0}_{3\times 9} & \mathbf{0}_3 \end{bmatrix} \tag{138}$$

$$\frac{\partial \tilde{\mathbf{z}}_3}{\partial {}^G\tilde{\mathbf{p}}_{fs}} = \mathbf{0}_3 \tag{139}$$

$$\frac{\partial \tilde{\mathbf{z}}_3}{\partial \tilde{\mathbf{x}}_T^{(2)}} = \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \end{bmatrix} \tag{140}$$

$$\frac{\partial \tilde{\mathbf{z}}_3}{\partial {}^T\tilde{\mathbf{p}}_{ft}} = \mathbf{0}_3 \tag{141}$$

5.2 State Transition Matrix

The total system state transition matrix can be written as:

$$\mathbf{\Phi}^{(2)} = \begin{bmatrix} \mathbf{\Phi}_I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Phi}_{fs} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Phi}_T^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{\Phi}_{ft} \end{bmatrix} \tag{142}$$

Note that $\mathbf{\Phi}_I$, $\mathbf{\Phi}_{fs}$ and $\mathbf{\Phi}_{ft}$ are the same as previous section. We still need to compute the target state transition matrix $\mathbf{\Phi}^{(2)}$ based on motion model 2. The linearized target motion system based on motion model 2 can be written as:

$$\dot{\tilde{x}}_T^{(2)} = \begin{bmatrix} \delta\dot{\boldsymbol{\theta}}_T \\ {}^T\dot{\tilde{\boldsymbol{\omega}}} \\ {}^G\dot{\tilde{\mathbf{p}}}_T \\ {}^T\dot{\tilde{\mathbf{v}}}_T \end{bmatrix} \simeq \begin{bmatrix} -\lfloor {}^T\hat{\boldsymbol{\omega}} \rfloor & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -{}^G_T\hat{\mathbf{R}} \lfloor {}^T\hat{\mathbf{v}} \rfloor & \mathbf{0}_3 & \mathbf{0}_3 & {}^G_T\hat{\mathbf{R}} \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{\theta}_T \\ {}^T\tilde{\boldsymbol{\omega}} \\ {}^G\tilde{\mathbf{p}}_T \\ {}^T\tilde{\mathbf{v}}_T \end{bmatrix} + \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{n}_{t\omega} \\ \mathbf{n}_{tv} \end{bmatrix} \tag{143}$$

From the linearized system, we can get the state transition $\mathbf{\Phi}_T^{(2)}$ evolution as:

$$\dot{\mathbf{\Phi}}_T^{(2)} = \begin{bmatrix} -\lfloor {}^T\hat{\boldsymbol{\omega}} \rfloor & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -{}^G_T\hat{\mathbf{R}} \lfloor {}^T\hat{\mathbf{v}} \rfloor & \mathbf{0}_3 & \mathbf{0}_3 & {}^G_T\hat{\mathbf{R}} \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{T11} & \mathbf{\Phi}_{T12} & \mathbf{\Phi}_{T13} & \mathbf{\Phi}_{T14} \\ \mathbf{\Phi}_{T21} & \mathbf{\Phi}_{T22} & \mathbf{\Phi}_{T23} & \mathbf{\Phi}_{T24} \\ \mathbf{\Phi}_{T31} & \mathbf{\Phi}_{T32} & \mathbf{\Phi}_{T33} & \mathbf{\Phi}_{T34} \\ \mathbf{\Phi}_{T41} & \mathbf{\Phi}_{T42} & \mathbf{\Phi}_{T43} & \mathbf{\Phi}_{T44} \end{bmatrix} \tag{144}$$

Hence, the target state transition matrix from t_0 to t_k can be solved as:

$$\mathbf{\Phi}_T^{(2)} = \begin{bmatrix} \mathbf{\Phi}_{T11} & \mathbf{\Phi}_{T12} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{\Phi}_{T31} & \mathbf{\Phi}_{T32} & \mathbf{I}_3 & \mathbf{\Phi}_{T34} \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{145}$$

where we have:

$$\mathbf{\Phi}_{T11} = {}^{T_k}_{T_0}\mathbf{R} \tag{146}$$

$$\mathbf{\Phi}_{T12} = \int_{t_0}^{t} {}^{T_k}_{T_\tau}\mathbf{R} d\tau \tag{147}$$

$$\dot{\mathbf{\Phi}}_{T31} = -\lfloor {}^{G}\mathbf{v}(t) \rfloor {}^{G}_{T_0}\mathbf{R} \tag{148}$$

$$\dot{\mathbf{\Phi}}_{T32} = -\lfloor {}^{G}\mathbf{v}(t) \rfloor \int_{t_0}^{t} {}^{G}_{T_\tau}\mathbf{R} d\tau \tag{149}$$

$$\mathbf{\Phi}_{T31} = -\lfloor {}^{G}\hat{\mathbf{p}}_{T_k} - {}^{G}\hat{\mathbf{p}}_{T_0} \rfloor {}^{G}_{T_0}\hat{\mathbf{R}} \tag{150}$$

$$\mathbf{\Phi}_{T32} = -\int_{t_0}^{t} \lfloor {}^{G}\mathbf{v}(s) \rfloor \int_{t_0}^{s} {}^{G}_{T_\tau}\hat{\mathbf{R}} d\tau ds \tag{151}$$

$$\mathbf{\Phi}_{T34} = \int_{t_0}^{t} {}^{G}_{T_\tau}\mathbf{R} d\tau \tag{152}$$

Therefore, the total state transition matrix can be written as:

$$\mathbf{\Phi}_T^{(2)} = \begin{bmatrix} {}^{T_k}_{T_0}\hat{\mathbf{R}} & \int_{t_0}^{t} {}^{T_k}_{T_\tau}\mathbf{R} d\tau & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^{G}\hat{\mathbf{p}}_{T_k} - {}^{G}\hat{\mathbf{p}}_{T_0} \rfloor {}^{G}_{T_0}\hat{\mathbf{R}} & -\int_{t_0}^{t} \lfloor {}^{G}\mathbf{v}(s) \rfloor \int_{t_0}^{s} {}^{G}_{T_\tau}\hat{\mathbf{R}} d\tau ds & \mathbf{I}_3 & \int_{t_0}^{t} {}^{G}_{T_\tau}\mathbf{R} d\tau \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{153}$$

5.3 Observability Matrix

With abuse of notation, the k'th block of observability matrix can be written as:

$$\mathbf{M}_k^{(2)} = \mathbf{H}_{\mathbf{x}k}\mathbf{\Phi}^{(2)}(k, 0) \tag{154}$$

$$= \mathbf{H}_{\mathbf{x}k} \begin{bmatrix} \mathbf{\Phi}_I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Phi}_{fs} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Phi}_T^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{\Phi}_{ft} \end{bmatrix} \tag{155}$$

$$= \begin{bmatrix} \mathbf{H}_{C1}{}^{I}_{G}\hat{\mathbf{R}} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{H}_{C2}{}^{I}_{G}\hat{\mathbf{R}} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}_{C3}{}^{I}_{G}\hat{\mathbf{R}} \end{bmatrix} \times$$

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} & \mathbf{M}_{14} & \mathbf{M}_{15} & \mathbf{M}_{16} & \mathbf{M}_{17} & \mathbf{M}_{18} & \mathbf{M}_{19} & \mathbf{M}_{1,10} & \mathbf{M}_{1,11} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} & \mathbf{M}_{24} & \mathbf{M}_{25} & \mathbf{M}_{26} & \mathbf{M}_{27} & \mathbf{M}_{28} & \mathbf{M}_{29} & \mathbf{M}_{2,10} & \mathbf{M}_{2,11} \\ \mathbf{M}_{31} & \mathbf{M}_{32} & \mathbf{M}_{33} & \mathbf{M}_{34} & \mathbf{M}_{35} & \mathbf{M}_{36} & \mathbf{M}_{37} & \mathbf{M}_{38} & \mathbf{M}_{39} & \mathbf{M}_{3,10} & \mathbf{M}_{3,11} \end{bmatrix} \tag{156}$$

Then, we can solve this matrix. The first row of this matrix can be denoted as:

$$\mathbf{M}_{11} = \lfloor {}^{G}\hat{\mathbf{p}}_{fs} - {}^{G}\hat{\mathbf{p}}_{I_k} \rfloor {}^{G}_{I_k}\hat{\mathbf{R}}\mathbf{\Phi}_{I11} - \mathbf{\Phi}_{I51} = \lfloor {}^{G}\hat{\mathbf{p}}_{fs} - {}^{G}\hat{\mathbf{p}}_{I_0} - {}^{G}\hat{\mathbf{v}}_{I_0}\delta t_k + \frac{1}{2}{}^{G}\mathbf{g}\delta t_k^2 \rfloor {}^{G}_{I_0}\hat{\mathbf{R}} \tag{157}$$

$$\mathbf{M}_{12} = \lfloor {}^{G}\hat{\mathbf{p}}_{fs} - {}^{G}\hat{\mathbf{p}}_{I_k} \rfloor {}^{G}_{I_k}\hat{\mathbf{R}}\mathbf{\Phi}_{I12} - \mathbf{\Phi}_{I52} \tag{158}$$

$$\mathbf{M}_{13} = -\mathbf{\Phi}_{I53} = -\mathbf{I}_3\delta t_k \tag{159}$$

$$\mathbf{M}_{14} = -\mathbf{\Phi}_{I54} \tag{160}$$

$$\mathbf{M}_{15} = -\mathbf{I}_3 \tag{161}$$

$$\mathbf{M}_{16} = \mathbf{I}_3 \tag{162}$$

$$\mathbf{M}_{17} = \mathbf{M}_{18} = \mathbf{M}_{19} = \mathbf{M}_{1,10} = \mathbf{M}_{1,11} = \mathbf{0}_3 \tag{163}$$

The second row of this matrix can be described as:

$$\mathbf{M}_{21} = \lfloor {}^G_{T_k}\hat{\mathbf{R}}^T\hat{\mathbf{p}}_{ft} - {}^G\hat{\mathbf{p}}_{I_k} + {}^G\hat{\mathbf{p}}_{T_k}\rfloor {}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Phi}_{I11} - \mathbf{\Phi}_{I51} \tag{164}$$

$$= \lfloor {}^G_{T_k}\hat{\mathbf{R}}^T\hat{\mathbf{p}}_{ft} + {}^G\hat{\mathbf{p}}_{T_k} - {}^G\hat{\mathbf{p}}_{I_0} - {}^G\mathbf{v}_{I_0}\delta t_k + \frac{1}{2}{}^G\mathbf{g}\delta t_k^2\rfloor {}^G_{I_0}\mathbf{R} \tag{165}$$

$$\mathbf{M}_{22} = \lfloor {}^G_{T_k}\hat{\mathbf{R}}^T\hat{\mathbf{p}}_{ft} - {}^G\hat{\mathbf{p}}_{I_k} + {}^G\hat{\mathbf{p}}_{T_k}\rfloor {}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Phi}_{I12} - \mathbf{\Phi}_{I52} \tag{166}$$

$$\mathbf{M}_{23} = -\mathbf{\Phi}_{I53} = -\mathbf{I}_3\delta t_k \tag{167}$$

$$\mathbf{M}_{24} = -\mathbf{\Phi}_{I54} \tag{168}$$

$$\mathbf{M}_{25} = -\mathbf{I}_3 \tag{169}$$

$$\mathbf{M}_{26} = \mathbf{0}_3 \tag{170}$$

$$\mathbf{M}_{27} = -{}^G_{T_k}\hat{\mathbf{R}}\lfloor {}^T\hat{\mathbf{p}}_{ft}\rfloor\mathbf{\Phi}_{T11} + \mathbf{\Phi}_{T31} = \lfloor -{}^G_{T_k}\hat{\mathbf{R}}^T\hat{\mathbf{p}}_{ft} - {}^G\hat{\mathbf{p}}_{T_k} + {}^G\hat{\mathbf{p}}_{T_0}\rfloor {}^G_{T_0}\hat{\mathbf{R}} \tag{171}$$

$$\mathbf{M}_{28} = -{}^G_{T_k}\hat{\mathbf{R}}\lfloor {}^T\hat{\mathbf{p}}_{ft}\rfloor\mathbf{\Phi}_{T12} + \mathbf{\Phi}_{T32} \tag{172}$$

$$\mathbf{M}_{29} = \mathbf{I}_3 \tag{173}$$

$$\mathbf{M}_{2,10} = \mathbf{\Phi}_{T34} = \int_{t_0}^{t} {}^G_{T_\tau}\mathbf{R}\mathbf{d}\tau \tag{174}$$

$$\mathbf{M}_{2,11} = {}^G_{T_k}\hat{\mathbf{R}} \tag{175}$$

The third row of this matrix can be described as:

$$\mathbf{M}_{31} = \lfloor {}^G\hat{\mathbf{p}}_{T_k} - {}^G\hat{\mathbf{p}}_{I_k}\rfloor {}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Phi}_{I11} - \mathbf{\Phi}_{I51} = \lfloor {}^G\hat{\mathbf{p}}_{T_k} - {}^G\hat{\mathbf{p}}_{I_0} - {}^G\hat{\mathbf{v}}_{I_0} + \frac{1}{2}{}^G\mathbf{g}\delta t_k^2\rfloor {}^G_{I_0}\mathbf{R} \tag{176}$$

$$\mathbf{M}_{32} = \lfloor {}^G\hat{\mathbf{p}}_{T_k} - {}^G\hat{\mathbf{p}}_{I_k}\rfloor {}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Phi}_{I12} - \mathbf{\Phi}_{I52} \tag{177}$$

$$\mathbf{M}_{33} = -\mathbf{\Phi}_{I53} = -\mathbf{I}_3\delta t_k \tag{178}$$

$$\mathbf{M}_{34} = -\mathbf{\Phi}_{I54} \tag{179}$$

$$\mathbf{M}_{35} = -\mathbf{I}_3 \tag{180}$$

$$\mathbf{M}_{36} = \mathbf{0}_3 \tag{181}$$

$$\mathbf{M}_{37} = -\mathbf{\Phi}_{T31} = \lfloor {}^G\hat{\mathbf{p}}_{T_k} - {}^G\hat{\mathbf{p}}_{T_0}\rfloor {}^G_{T_0}\hat{\mathbf{R}} \tag{182}$$

$$\mathbf{M}_{37} = -\mathbf{\Phi}_{T32} \tag{183}$$

$$\mathbf{M}_{39} = -\mathbf{I}_3 \tag{184}$$

$$\mathbf{M}_{3,10} = \mathbf{\Phi}_{T34} \tag{185}$$

$$\mathbf{M}_{3,11} = \mathbf{0}_3 \tag{186}$$

Therefore, we can rewrite the equation as:

$$\mathbf{M}_k^{(2)} = \mathbf{H}_{\mathbf{x}k}\mathbf{\Phi}^{(2)}(k,0) \tag{187}$$

$$= \begin{bmatrix} \mathbf{H}_{C1}{}^I_G\mathbf{R} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{H}_{C2}{}^I_G\hat{\mathbf{R}} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}_{C3}{}^I_G\hat{\mathbf{R}} \end{bmatrix} \times$$
$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & -\mathbf{I}_3\delta t_k & \mathbf{M}_{14} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{M}_{21} & \mathbf{M}_{22} & -\mathbf{I}_3\delta t_k & \mathbf{M}_{24} & -\mathbf{I}_3 & \mathbf{0}_3 & \lfloor -{}^G_{T_k}\hat{\mathbf{R}}^T\hat{\mathbf{p}}_{ft} - {}^G\hat{\mathbf{p}}_{T_k} + {}^G\hat{\mathbf{p}}_{T_0}\rfloor {}^G_{T_0}\hat{\mathbf{R}} & \mathbf{M}_{28} & \mathbf{I}_3 & \mathbf{M}_{2,10} & {}^G_{T_k}\hat{\mathbf{R}} \\ \mathbf{M}_{31} & \mathbf{0}_3 & -\mathbf{I}_3\delta t_k & \mathbf{M}_{34} & -\mathbf{I}_3 & \mathbf{0}_3 & \lfloor {}^G\hat{\mathbf{p}}_{T_k} - {}^G\hat{\mathbf{p}}_{T_0}\rfloor {}^G_{T_0}\hat{\mathbf{R}} & \mathbf{M}_{38} & -\mathbf{I}_3 & \mathbf{M}_{3,10} & \mathbf{0}_3 \end{bmatrix} \tag{188}$$

Based on the constant local velocity assumption, the ${}^{T}\boldsymbol{\omega}$ and ${}^{T}\mathbf{v}_T$ are constant. Therefore, we have:

$$\boldsymbol{\Phi}_{T12}\lfloor {}^{T}\hat{\boldsymbol{\omega}}\rfloor = \int_{t_0}^{t} {}^{T_k}_{T_\tau}\mathbf{R}\mathbf{d}\tau \lfloor {}^{T}\hat{\boldsymbol{\omega}}\rfloor = \int_{t_0}^{t} {}^{T_k}_{T_\tau}\mathbf{R}\lfloor {}^{T_\tau}\boldsymbol{\omega}\rfloor \mathbf{d}\tau = {}^{T_k}_{T_k}\hat{\mathbf{R}} - {}^{T_k}_{T_0}\hat{\mathbf{R}} = \mathbf{I}_3 - {}^{T_k}_{T_0}\hat{\mathbf{R}} \tag{189}$$

$$\boldsymbol{\Phi}_{T32}\lfloor {}^{T}\hat{\boldsymbol{\omega}}\rfloor = -\int_{t_0}^{t} \lfloor {}^{G}\mathbf{v}(s)\rfloor \int_{t_0}^{s} {}^{G}_{T_\tau}\mathbf{R}\mathbf{d}\tau \mathbf{d}s \lfloor {}^{T}\boldsymbol{\omega}\rfloor \tag{190}$$

$$= -\int_{t_0}^{t} \lfloor {}^{G}\mathbf{v}(s)\rfloor \left({}^{G}_{T_s}\hat{\mathbf{R}} - {}^{G}_{T_0}\hat{\mathbf{R}} \right) \mathbf{d}s \tag{191}$$

$$= -\int_{t_0}^{t} \lfloor {}^{G}\mathbf{v}(s)\rfloor {}^{G}_{T_s}\hat{\mathbf{R}}\mathbf{d}s - \lfloor {}^{G}\hat{\mathbf{p}}_{T_k} - {}^{G}\hat{\mathbf{p}}_{T_0}\rfloor {}^{G}_{T_0}\hat{\mathbf{R}} \tag{192}$$

$$= -\int_{t_0}^{t} {}^{G}_{T_s}\hat{\mathbf{R}}\lfloor {}^{T}\mathbf{v}\rfloor \mathbf{d}s - \lfloor {}^{G}\hat{\mathbf{p}}_{T_k} - {}^{G}\hat{\mathbf{p}}_{T_0}\rfloor {}^{G}_{T_0}\hat{\mathbf{R}} \tag{193}$$

$$\boldsymbol{\Phi}_{T34}\lfloor {}^{T}\hat{\boldsymbol{\omega}}\rfloor = \int_{t_0}^{t} {}^{G}_{T_\tau}\mathbf{R}\mathbf{d}\tau \lfloor {}^{T}\hat{\boldsymbol{\omega}}\rfloor = \int_{t_0}^{t} {}^{G}_{T_\tau}\mathbf{R}\lfloor {}^{T_\tau}\boldsymbol{\omega}\rfloor \mathbf{d}\tau = {}^{G}_{T_k}\hat{\mathbf{R}} - {}^{G}_{T_0}\hat{\mathbf{R}} \tag{194}$$

The unobservable directions $\mathbf{N}^{(2)}$ span the right null space of the observability matrix $\mathbf{M}^{(2)}$, that is: $\mathbf{M}^{(2)}\mathbf{N}^{(2)} = \mathbf{0}$. Based on the derivation of the observability matrix, we can have the unobservable directions as:

$$\mathbf{N}^{(2)} = \begin{bmatrix} \mathbf{N}_1^{(2)} & \mathbf{N}_{2:4}^{(2)} & \mathbf{N}_{{}^{T}_{G}\mathbf{R}}^{(2)} \end{bmatrix} = \begin{bmatrix} {}^{I_0}_{G}\hat{\mathbf{R}}{}^{G}\mathbf{g} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^{G}\hat{\mathbf{v}}_{I_0}\rfloor {}^{G}\mathbf{g} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^{G}\hat{\mathbf{p}}_{I_0}\rfloor {}^{G}\mathbf{g} & \mathbf{I}_3 & \mathbf{0}_3 \\ -\lfloor {}^{G}\hat{\mathbf{p}}_{fs}\rfloor {}^{G}\mathbf{g} & \mathbf{I}_3 & \mathbf{0}_3 \\ -{}^{T_0}_{I_0}\hat{\mathbf{R}}{}^{G}\mathbf{g} & \mathbf{0}_3 & \mathbf{I}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \lfloor {}^{T}\hat{\boldsymbol{\omega}}\rfloor \\ -\lfloor {}^{G}\hat{\mathbf{p}}_{T_0}\rfloor {}^{G}\mathbf{g} & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \lfloor {}^{T}\hat{\mathbf{v}}\rfloor \\ \mathbf{0}_3 & \mathbf{0}_3 & \lfloor {}^{T}\hat{\mathbf{p}}_{ft}\rfloor \end{bmatrix} \tag{195}$$

If without the representative point ${}^{G}\mathbf{p}_T$ measurement, there will be one more unobservable directions regarding to the representative point position of the target as:

$$\mathbf{N}_{{}^{G}\mathbf{p}_T}^{(2)} = \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \left({}^{G}_{T_0}\hat{\mathbf{R}}\right)^{\top} & \left(\lfloor {}^{I}\hat{\boldsymbol{\omega}}\rfloor\right)^{\top} & -\mathbf{I}_3 \end{bmatrix}^{\top} \tag{196}$$

5.4 Geometrical Interpretation - Verification of $\mathbf{N}_{{}^{T}_{G}\mathbf{R}}^{(2)}$

If we disturb the target orientation by $\delta\boldsymbol{\phi}$, then we will have the disturbed state vector as:

$${}^{T'}_{G}\mathbf{R} = {}^{T'}_{T}\mathbf{R}{}^{T}_{G}\mathbf{R} \simeq (\mathbf{I}_3 - \lfloor\delta\boldsymbol{\phi}\rfloor)\,{}^{T}_{G}\mathbf{R} \tag{197}$$

$${}^{T'}\omega = {}^{T'}_{T}\mathbf{R}{}^{T}\omega \simeq (\mathbf{I}_3 - \lfloor\delta\boldsymbol{\phi}\rfloor)\,{}^{T}\boldsymbol{\omega} = {}^{T}\boldsymbol{\omega} + \lfloor {}^{T}\boldsymbol{\omega}\rfloor\delta\boldsymbol{\phi} \tag{198}$$

$${}^{G}\mathbf{p}_{T'} = {}^{G}\mathbf{p}_{T} \tag{199}$$

$${}^{T'}\mathbf{v}_{T'} = {}^{T'}_{T}\mathbf{R}{}^{T}\mathbf{v}_T \simeq (\mathbf{I}_3 - \lfloor\delta\boldsymbol{\phi}\rfloor)\,{}^{T}\mathbf{v}_T = {}^{T}\mathbf{v}_T + \lfloor {}^{T}\mathbf{v}_T\rfloor\delta\boldsymbol{\phi} \tag{200}$$

$${}^{T'}\mathbf{p}_{ft} = {}^{T'}_{T}\mathbf{R}{}^{T}\mathbf{p}_{ft} \simeq (\mathbf{I}_3 - \lfloor\delta\boldsymbol{\phi}\rfloor)\,{}^{T}\mathbf{p}_{ft} = {}^{T}\mathbf{p}_{ft} + \lfloor {}^{T}\mathbf{p}_{ft}\rfloor\delta\boldsymbol{\phi} \tag{201}$$

Then, for the target feature measurements and target representative point measurement, we have:

$$
\begin{aligned}
{}^{I}\mathbf{p}_{ft'} &= {}^{I}_{G}\mathbf{R}\left({}^{G}_{T'}\mathbf{R}^{T'}\mathbf{p}_{ft} - {}^{G}\mathbf{p}_{I} + {}^{G}\mathbf{p}_{T'}\right) = {}^{I}_{G}\mathbf{R}\left({}^{G}_{T}\mathbf{R}^{T}_{T'}\mathbf{R}^{T'}_{T}\mathbf{R}^{T}\mathbf{p}_{ft} - {}^{G}\mathbf{p}_{I} + {}^{G}\mathbf{p}_{T}\right) && (202)\\
&= {}^{I}\mathbf{p}_{ft} && (203)\\
{}^{I}\mathbf{p}_{T'} &= {}^{I}_{G}\mathbf{R}\left({}^{G}\mathbf{p}_{T'} - {}^{G}\mathbf{p}_{I}\right) = {}^{I}_{G}\mathbf{R}\left({}^{G}\mathbf{p}_{T} - {}^{G}\mathbf{p}_{I}\right) && (204)\\
&= {}^{I}\mathbf{p}_{T} && (205)
\end{aligned}
$$

Hence, after the orientation disturbance, the system will still have the same target feature and target representative point measurements. Since we assume $\delta\boldsymbol{\phi}$ is a small perturbation, we can linearize the disturbed system state vector at $\hat{\mathbf{x}}$, then the related error states can be written as:

$$
\tilde{\mathbf{x}}' = \begin{bmatrix} \tilde{\mathbf{x}}'_{I} \\ {}^{G}\tilde{\mathbf{p}}'_{fs} \\ \delta\theta_{T'} \\ {}^{T'}\tilde{\boldsymbol{\omega}} \\ {}^{G}\tilde{\mathbf{p}}_{T'} \\ {}^{T'}\tilde{\mathbf{v}}_{T'} \\ {}^{T'}\tilde{\mathbf{p}}_{ft} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{x}}_{I} \\ {}^{G}\tilde{\mathbf{p}}_{fs} \\ \delta\theta_{T} + \delta\boldsymbol{\phi} \\ {}^{T}\tilde{\boldsymbol{\omega}} + \lfloor {}^{T}\hat{\boldsymbol{\omega}} \rfloor \delta\boldsymbol{\phi} \\ {}^{G}\tilde{\mathbf{p}}_{T} \\ {}^{T}\mathbf{v}_{T} + \lfloor {}^{T}\hat{\mathbf{v}}_{T} \rfloor \delta\boldsymbol{\phi} \\ {}^{T}\tilde{\mathbf{p}}_{ft} + \lfloor {}^{T}\hat{\mathbf{p}}_{ft} \rfloor \delta\boldsymbol{\phi} \end{bmatrix} = \tilde{\mathbf{x}} + \begin{bmatrix} \mathbf{0}_{18\times 3} \\ \mathbf{I}_{3} \\ \lfloor {}^{T}\hat{\boldsymbol{\omega}} \rfloor \\ \mathbf{0}_{3} \\ \lfloor {}^{T}\hat{\mathbf{v}}_{T} \rfloor \\ \lfloor {}^{T}\hat{\mathbf{p}}_{ft} \rfloor \end{bmatrix} \delta\boldsymbol{\phi} = \tilde{\mathbf{x}} + \mathbf{N}^{(2)}_{{}^{T}_{G}\mathbf{R}}\delta\boldsymbol{\theta} \quad (206)
$$

5.5 Geometrical Interpretation - Verification of $\mathbf{N}^{(2)}_{{}^{G}\mathbf{p}_{T}}$

Similarly, we can verify unobservable directions related to the target representative point position $\mathbf{N}_{{}^{G}\mathbf{p}_{T}}$. We assume there is a small disturbance to representative position as $\delta\mathbf{p}$, then we can have the disturbed state parameters as:

$$
\begin{aligned}
{}^{T'}_{G}\mathbf{R} &= {}^{T}_{G}\mathbf{R} && (207)\\
{}^{T'}\boldsymbol{\omega} &= {}^{T}\boldsymbol{\omega} && (208)\\
{}^{G}\mathbf{p}_{T'} &= {}^{G}\mathbf{p}_{T} + {}^{G}_{T}\mathbf{R}\delta\mathbf{p} \Rightarrow {}^{T}\mathbf{p}_{T'} = {}^{T}_{G}\mathbf{R}\left({}^{G}\mathbf{p}_{T'} - {}^{G}\mathbf{p}_{T}\right) = \delta\mathbf{p} && (209)\\
{}^{T'}\mathbf{v}_{T'} &= {}^{T'}_{T}\mathbf{R}\left({}^{T}\mathbf{v}_{T} + \lfloor {}^{T}\hat{\boldsymbol{\omega}} \rfloor \delta\mathbf{p}\right) && (210)\\
{}^{T'}\mathbf{p}_{ft} &= {}^{T'}_{T}\mathbf{R}\left({}^{T}\mathbf{p}_{ft} - {}^{T}\mathbf{p}_{T'}\right) = {}^{T}\mathbf{p}_{ft} - \delta\mathbf{p} && (211)
\end{aligned}
$$

Therefore, the related target feature in IMU frame can be written as:

$$
\begin{aligned}
{}^{I}\mathbf{p}_{ft'} &= {}^{I}_{G}\mathbf{R}\left({}^{G}_{T'}\mathbf{R}^{T'}\mathbf{p}_{ft} - {}^{G}\mathbf{p}_{I} + {}^{G}\mathbf{p}_{T'}\right) = {}^{I}_{G}\mathbf{R}\left({}^{G}_{T}\mathbf{R}\left({}^{T}\mathbf{p}_{ft} - \delta\mathbf{p}\right) - {}^{G}\mathbf{p}_{I} + {}^{G}\mathbf{p}_{T} + {}^{G}_{T}\mathbf{R}\delta\mathbf{p}\right) && (212)\\
&= {}^{I}\mathbf{p}_{ft} && (213)
\end{aligned}
$$

That means, the system will have the the same target feature measurements even given the distance disturbance. Hence, with the disturbance to target position, we can write the disturbed error states as:

$$
\tilde{\mathbf{x}}' = \begin{bmatrix} \tilde{\mathbf{x}}'_{I} \\ {}^{G}\tilde{\mathbf{p}}'_{fs} \\ \delta\theta_{T'} \\ {}^{T'}\tilde{\boldsymbol{\omega}} \\ {}^{G}\tilde{\mathbf{p}}_{T'} \\ {}^{T'}\tilde{\mathbf{v}}_{T'} \\ {}^{T'}\tilde{\mathbf{p}}_{ft} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{x}}_{I} \\ {}^{G}\tilde{\mathbf{p}}_{fs} \\ \delta\theta_{T} \\ {}^{T}\tilde{\boldsymbol{\omega}} \\ {}^{G}\tilde{\mathbf{p}}_{T} + {}^{G}_{T}\hat{\mathbf{R}}\delta\mathbf{p} \\ {}^{T}\tilde{\mathbf{v}}_{T} + \lfloor {}^{T}\hat{\boldsymbol{\omega}} \rfloor \delta\mathbf{p} \\ {}^{T}\tilde{\mathbf{p}}_{ft} - \delta\mathbf{p} \end{bmatrix} = \tilde{\mathbf{x}} + \begin{bmatrix} \mathbf{0}_{18\times 3} \\ \mathbf{0}_{3} \\ \mathbf{0}_{3} \\ {}^{G}_{T}\hat{\mathbf{R}} \\ \lfloor {}^{T}\hat{\boldsymbol{\omega}} \rfloor \\ -\mathbf{I}_{3} \end{bmatrix} \delta\mathbf{p} = \tilde{\mathbf{x}} + \mathbf{N}^{(2)}_{{}^{G}\mathbf{p}_{T}}\delta\mathbf{p} \quad (214)
$$

6 Observability Analysis - Motion Model 3

6.1 Measurement Jacobians

Based on motion model 3, the measurement Jacobians can be written as:

$$\mathbf{H}_{\mathbf{x}} = \begin{bmatrix} \mathbf{H}_1 \\ \mathbf{H}_2 \\ \mathbf{H}_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}} \\ \frac{\partial \tilde{\mathbf{z}}_2}{\partial \tilde{\mathbf{x}}} \\ \frac{\partial \tilde{\mathbf{z}}_3}{\partial \tilde{\mathbf{x}}} \end{bmatrix} \tag{215}$$

The Jacobians of the measurement to the environmental feature $\mathbf{H}_1$ can be written as:

$$\mathbf{H}_1 = \frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}} = \begin{bmatrix} \frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}_I} & \frac{\partial \tilde{\mathbf{z}}_1}{\partial {}^G\tilde{\mathbf{p}}_{fs}} & \frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}_T^{(3)}} & \frac{\partial \tilde{\mathbf{z}}_1}{\partial {}^T\tilde{\mathbf{p}}_{ft}} \end{bmatrix} \tag{216}$$

where we have:

$$\mathbf{H}_{C1} = \frac{1}{{}^I\hat{z}_{fs}^2} \begin{bmatrix} {}^I\hat{z}_{fs} & 0 & -{}^I\hat{x}_{fs} \\ 0 & {}^I\hat{z}_{fs} & -{}^I\hat{y}_{fs} \end{bmatrix} \tag{217}$$

$$\frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}_I} = \mathbf{H}_{C1} {}^I_G\hat{\mathbf{R}} \begin{bmatrix} \lfloor {}^G\hat{\mathbf{p}}_{fs} - {}^G\hat{\mathbf{p}}_I \rfloor {}^G_I\hat{\mathbf{R}} & \mathbf{0}_{3\times 9} & -\mathbf{I}_3 \end{bmatrix} \tag{218}$$

$$\frac{\partial \tilde{\mathbf{z}}_1}{\partial {}^G\tilde{\mathbf{p}}_{fs}} = \mathbf{H}_{C1} {}^I_G\hat{\mathbf{R}} \mathbf{I}_3 \tag{219}$$

$$\frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}_T^{(3)}} = \mathbf{0}_{3\times 9} \tag{220}$$

$$\frac{\partial \tilde{\mathbf{z}}_1}{\partial {}^T\tilde{\mathbf{p}}_{ft}} = \mathbf{0}_3 \tag{221}$$

The target feature measurement Jacobians $\mathbf{H}_2$ can be written as:

$$\mathbf{H}_2 = \frac{\partial \tilde{\mathbf{z}}_2}{\partial \tilde{\mathbf{x}}} = \begin{bmatrix} \frac{\partial \tilde{\mathbf{z}}_2}{\partial \tilde{\mathbf{x}}_I} & \frac{\partial \tilde{\mathbf{z}}_2}{\partial {}^G\tilde{\mathbf{p}}_{fs}} & \frac{\partial \tilde{\mathbf{z}}_2}{\partial \tilde{\mathbf{x}}_T^{(3)}} & \frac{\partial \tilde{\mathbf{z}}_2}{\partial {}^T\tilde{\mathbf{p}}_{ft}} \end{bmatrix} \tag{222}$$

where we have:

$$\mathbf{H}_{C2} = \frac{1}{{}^I\hat{z}_{ft}^2} \begin{bmatrix} {}^I\hat{z}_{ft} & 0 & -{}^I\hat{x}_{ft} \\ 0 & {}^I\hat{z}_{ft} & -{}^I\hat{y}_{ft} \end{bmatrix} \tag{223}$$

$$\frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}_I} = \mathbf{H}_{C2} {}^I_G\hat{\mathbf{R}} \begin{bmatrix} \lfloor \left({}^T_G\mathbf{R}^\top {}^T\hat{\mathbf{p}}_{ft} - \left({}^G\hat{\mathbf{p}}_I - {}^G\hat{\mathbf{p}}_T \right) \right) \rfloor {}^G_I\hat{\mathbf{R}} & \mathbf{0}_{3\times 9} & -\mathbf{I}_3 \end{bmatrix} \tag{224}$$

$$\frac{\partial \tilde{\mathbf{z}}_2}{\partial {}^G\tilde{\mathbf{p}}_{fs}} = \mathbf{0}_3 \tag{225}$$

$$\frac{\partial \tilde{\mathbf{z}}_2}{\partial \tilde{\mathbf{x}}_T^{(3)}} = \mathbf{H}_{C2} {}^I_G\hat{\mathbf{R}} \begin{bmatrix} -{}^T_G\hat{\mathbf{R}}^\top \lfloor {}^T\hat{\mathbf{p}}_{ft} \rfloor & \mathbf{0}_{3\times 1} & \mathbf{I}_3 & \mathbf{0}_{3\times 2} \end{bmatrix} \tag{226}$$

$$\frac{\partial \tilde{\mathbf{z}}_2}{\partial {}^T\tilde{\mathbf{p}}_{ft}} = \mathbf{H}_{C2} {}^I_G\hat{\mathbf{R}} {}^T_G\hat{\mathbf{R}}^\top \tag{227}$$

The target representative point measurement Jacobians can be written as:

$$\mathbf{H}_3 = \frac{\partial \tilde{\mathbf{z}}_2}{\partial \tilde{\mathbf{x}}} = \begin{bmatrix} \frac{\partial \tilde{\mathbf{z}}_3}{\partial \tilde{\mathbf{x}}_I} & \frac{\partial \tilde{\mathbf{z}}_3}{\partial {}^G\tilde{\mathbf{p}}_{fs}} & \frac{\partial \tilde{\mathbf{z}}_3}{\partial \tilde{\mathbf{x}}_T^{(3)}} & \frac{\partial \tilde{\mathbf{z}}_3}{\partial {}^T\tilde{\mathbf{p}}_{ft}} \end{bmatrix} \tag{228}$$

where we have:

$$\mathbf{H}_{C3} = \frac{1}{{}^I\hat{z}_T^2}\begin{bmatrix} {}^I\hat{z}_T & 0 & -{}^I\hat{x}_T \\ 0 & {}^I\hat{z}_T & -{}^I\hat{y}_T \end{bmatrix} \tag{229}$$

$$\frac{\partial \tilde{\mathbf{z}}_1}{\partial \tilde{\mathbf{x}}_I} = \mathbf{H}_{C3}{}^I_G\hat{\mathbf{R}}\begin{bmatrix} \lfloor\left({}^G\hat{\mathbf{p}}_T - {}^G\hat{\mathbf{p}}_I\right)\rfloor{}^G_I\hat{\mathbf{R}} & \mathbf{0}_{3\times 9} & \mathbf{0}_3 \end{bmatrix} \tag{230}$$

$$\frac{\partial \tilde{\mathbf{z}}_3}{\partial {}^G\tilde{\mathbf{p}}_{fs}} = \mathbf{0}_3 \tag{231}$$

$$\frac{\partial \tilde{\mathbf{z}}_3}{\partial \tilde{\mathbf{x}}_T^{(3)}} = \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_{3\times 1} & \mathbf{I}_3 & \mathbf{0}_{3\times 2} \end{bmatrix} \tag{232}$$

$$\frac{\partial \tilde{\mathbf{z}}_3}{\partial {}^T\tilde{\mathbf{p}}_{ft}} = \mathbf{0}_3 \tag{233}$$

6.2 State Transition Matrix

The total system state transition matrix can be written as:

$$\mathbf{\Phi}^{(3)} = \begin{bmatrix} \mathbf{\Phi}_I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Phi}_{fs} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Phi}_T^{(3)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{\Phi}_{ft} \end{bmatrix} \tag{234}$$

Note that $\mathbf{\Phi}_I$, $\mathbf{\Phi}_{fs}$ and $\mathbf{\Phi}_{ft}$ are the same as previous sections. We still need to solve the target state transition matrix $\mathbf{\Phi}_T^{(3)}$. We first linearize the related state evolution functions based on model 3, and get:

$$\delta\dot{\boldsymbol{\theta}} = -\hat{\omega}_z\lfloor\mathbf{e}_3\rfloor\delta\boldsymbol{\theta} + \mathbf{e}_3\hat{\omega}_z + \begin{bmatrix} \mathbf{I}_2 \\ \mathbf{0}_{1\times 2} \end{bmatrix}\begin{bmatrix} n_{w_x} \\ n_{w_y} \end{bmatrix} \tag{235}$$

$${}^G\dot{\tilde{\mathbf{p}}}_T = -{}^T_G\hat{\mathbf{R}}^\top\lfloor\begin{bmatrix} \hat{v}_x \\ \hat{v}_y \\ 0 \end{bmatrix}\rfloor\delta\boldsymbol{\theta}_T + {}^T_G\hat{\mathbf{R}}^\top\begin{bmatrix} \tilde{v}_x \\ \tilde{v}_y \\ 0 \end{bmatrix} + {}^T_G\hat{\mathbf{R}}^\top\mathbf{e}_3 n_{vz} \tag{236}$$

The linearized system can be written as:

$$\dot{\tilde{x}}_T^{(3)} = \begin{bmatrix} \delta\dot{\boldsymbol{\theta}}_T \\ {}^T\dot{\tilde{\omega}}_z \\ {}^G\dot{\tilde{\mathbf{p}}}_T \\ {}^T\dot{\tilde{v}}_x \\ {}^T\dot{\tilde{v}}_y \end{bmatrix} = \begin{bmatrix} -\hat{\omega}_z\lfloor\mathbf{e}_3\rfloor & \mathbf{e}_3 & \mathbf{0}_3 & \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 0 & \mathbf{0}_{1\times 3} & 0 & 0 \\ -{}^T_G\hat{\mathbf{R}}^\top\lfloor\begin{bmatrix} \hat{v}_x \\ \hat{v}_y \\ 0 \end{bmatrix}\rfloor & \mathbf{0}_{3\times 1} & \mathbf{0}_3 & {}^T_G\hat{\mathbf{R}}^\top\mathbf{e}_1 & {}^T_G\hat{\mathbf{R}}^\top\mathbf{e}_2 \\ \mathbf{0}_{1\times 3} & 0 & \mathbf{0}_{1\times 3} & 0 & 0 \\ \mathbf{0}_{1\times 3} & 0 & \mathbf{0}_{1\times 3} & 0 & 0 \end{bmatrix}\begin{bmatrix} \delta\boldsymbol{\theta}_T \\ {}^T\tilde{\omega}_z \\ {}^G\tilde{\mathbf{p}}_T \\ {}^T\tilde{v}_x \\ {}^T\tilde{v}_y \end{bmatrix}$$
$$+ \begin{bmatrix} \begin{bmatrix} \mathbf{I}_2 \\ \mathbf{0}_{1\times 2} \end{bmatrix} & \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 2} & \mathbf{0}_1 & 1 & 0 & 0 \\ \mathbf{0}_{3\times 2} & \mathbf{0}_{3\times 1} & {}^T_G\hat{\mathbf{R}}^\top\mathbf{e}_3 & \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 2} & \mathbf{0}_1 & 0 & 1 & 0 \\ \mathbf{0}_{1\times 2} & \mathbf{0}_1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} n_{wx} \\ n_{wy} \\ n_{Twz} \\ n_z \\ n_{Tx} \\ n_{Ty} \end{bmatrix} \tag{237}$$

From the linearized system, we can get the state transition matrix $\mathbf{\Phi}_T^{(3)}$ evolution as:

$$\dot{\mathbf{\Phi}}_T^{(3)} = \begin{bmatrix} -\hat{\omega}_z \lfloor \mathbf{e}_3 \rfloor & \mathbf{e}_3 & \mathbf{0}_3 & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 0 & \mathbf{0}_{1\times3} & 0 & 0 \\ -{}^T_G\hat{\mathbf{R}}^\top \lfloor \begin{bmatrix} \hat{v}_x \\ \hat{v}_y \\ 0 \end{bmatrix} \rfloor & \mathbf{0}_{3\times1} & \mathbf{0}_3 & {}^T_G\hat{\mathbf{R}}^\top \mathbf{e}_1 & {}^T_G\hat{\mathbf{R}}^\top \mathbf{e}_2 \\ \mathbf{0}_{1\times3} & 0 & \mathbf{0}_{1\times3} & 0 & 0 \\ \mathbf{0}_{1\times3} & 0 & \mathbf{0}_{1\times3} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{11} & \mathbf{\Phi}_{12} & \mathbf{\Phi}_{13} & \mathbf{\Phi}_{14} & \mathbf{\Phi}_{15} \\ \mathbf{\Phi}_{21} & \mathbf{\Phi}_{22} & \mathbf{\Phi}_{23} & \mathbf{\Phi}_{24} & \mathbf{\Phi}_{25} \\ \mathbf{\Phi}_{31} & \mathbf{\Phi}_{32} & \mathbf{\Phi}_{33} & \mathbf{\Phi}_{34} & \mathbf{\Phi}_{35} \\ \mathbf{\Phi}_{41} & \mathbf{\Phi}_{42} & \mathbf{\Phi}_{43} & \mathbf{\Phi}_{44} & \mathbf{\Phi}_{45} \\ \mathbf{\Phi}_{51} & \mathbf{\Phi}_{52} & \mathbf{\Phi}_{53} & \mathbf{\Phi}_{54} & \mathbf{\Phi}_{55} \end{bmatrix} \tag{238}$$

Therefore, we can define the state transition matrix as:

$$\mathbf{\Phi}_T^{(3)} = \begin{bmatrix} \mathbf{\Phi}_{T11} & \mathbf{\Phi}_{T12} & \mathbf{0}_3 & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 & \mathbf{0}_{1\times3} & 0 & 0 \\ \mathbf{\Phi}_{T31} & \mathbf{\Phi}_{T32} & \mathbf{I}_3 & \mathbf{\Phi}_{T34} & \mathbf{\Phi}_{T35} \\ \mathbf{0}_{1\times3} & 0 & \mathbf{0}_{1\times3} & 1 & 0 \\ \mathbf{0}_{1\times3} & 0 & \mathbf{0}_{1\times3} & 0 & 1 \end{bmatrix} \tag{239}$$

where we have:

$$\mathbf{\Phi}_{T11} = {}^{T_k}_{T_0}\mathbf{R} \tag{240}$$

$$\mathbf{\Phi}_{T12} = \int_{t_0}^{t} {}^{T_k}_{T_\tau}\mathbf{R} d\tau \mathbf{e}_3 \tag{241}$$

$$\dot{\mathbf{\Phi}}_{T31} = -\lfloor {}^G\mathbf{v}_T(t) \rfloor {}^G_{T_0}\mathbf{R} \tag{242}$$

$$\dot{\mathbf{\Phi}}_{T32} = -\lfloor {}^G\mathbf{v}_T(t) \rfloor \int_{t_0}^{t} {}^G_{T_\tau}\mathbf{R} d\tau \mathbf{e}_3 \tag{243}$$

$$\mathbf{\Phi}_{T31} = -\lfloor {}^G\hat{\mathbf{p}}_{T_k} - {}^G\hat{\mathbf{p}}_{T_0} \rfloor {}^G_{T_0}\hat{\mathbf{R}} \tag{244}$$

$$\mathbf{\Phi}_{T32} = -\int_{t_0}^{t} \lfloor {}^G\mathbf{v}_T(s) \rfloor \int_{t_0}^{s} {}^G_{T_\tau}\hat{\mathbf{R}} d\tau ds \mathbf{e}_3 \tag{245}$$

$$\mathbf{\Phi}_{T34} = \int_{t_0}^{t} {}^G_{T_\tau}\mathbf{R} d\tau \mathbf{e}_1 \tag{246}$$

$$\mathbf{\Phi}_{T35} = \int_{t_0}^{t} {}^G_{T_\tau}\mathbf{R} d\tau \mathbf{e}_2 \tag{247}$$

Therefore, the target state transition matrix can be written as:

$$\mathbf{\Phi}_T^{(3)} = \begin{bmatrix} {}^{T_k}_{T_0}\hat{\mathbf{R}} & \int_{t_0}^{t} {}^{T_k}_{T_\tau}\mathbf{R} d\tau \mathbf{e}_3 & \mathbf{0}_3 & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 & \mathbf{0}_{1\times3} & 0 & 0 \\ -\lfloor {}^G\hat{\mathbf{p}}_{T_k} - {}^G\hat{\mathbf{p}}_{T_0} \rfloor {}^G_{T_0}\hat{\mathbf{R}} & -\int_{t_0}^{t} \lfloor {}^G\mathbf{v}_T(s) \rfloor \int_{t_0}^{s} {}^G_{T_\tau}\mathbf{R} d\tau ds \mathbf{e}_3 & \mathbf{I}_3 & \int_{t_0}^{t} {}^G_{T_\tau}\mathbf{R} d\tau \mathbf{e}_1 & \int_{t_0}^{t} {}^G_{T_\tau}\mathbf{R} d\tau \mathbf{e}_2 \\ \mathbf{0}_{1\times3} & 0 & \mathbf{0}_{1\times3} & 1 & 0 \\ \mathbf{0}_{1\times3} & 0 & \mathbf{0}_{1\times3} & 0 & 1 \end{bmatrix} \tag{248}$$

6.3 Observability Matrix

With abuse of notation, the k'th block of the observability matrix can be written as:

$$\mathbf{M}_k^{(3)} = \mathbf{H}_{\mathbf{x}k}\mathbf{\Phi}^{(3)}(k,0) \tag{249}$$

$$= \mathbf{H}_{\mathbf{x}k}\begin{bmatrix}\mathbf{\Phi}_I & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Phi}_{fs} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{\Phi}_T^{(3)} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{\Phi}_{ft}\end{bmatrix} \tag{250}$$

$$= \begin{bmatrix}\mathbf{H}_{C1}{}^I_G\mathbf{R} & \mathbf{0}_3 & \mathbf{0}_3\\ \mathbf{0}_3 & \mathbf{H}_{C2}{}^I_G\mathbf{R} & \mathbf{0}_3\\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}_{C3}{}^I_G\mathbf{R}\end{bmatrix}\times$$
$$\begin{bmatrix}\mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} & \mathbf{M}_{14} & \mathbf{M}_{15} & \mathbf{M}_{16} & \mathbf{M}_{17} & \mathbf{M}_{18} & \mathbf{M}_{19} & \mathbf{M}_{1,10} & \mathbf{M}_{1,11} & \mathbf{M}_{1,12}\\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} & \mathbf{M}_{24} & \mathbf{M}_{25} & \mathbf{M}_{26} & \mathbf{M}_{27} & \mathbf{M}_{28} & \mathbf{M}_{29} & \mathbf{M}_{2,10} & \mathbf{M}_{2,11} & \mathbf{M}_{2,12}\\ \mathbf{M}_{31} & \mathbf{M}_{32} & \mathbf{M}_{33} & \mathbf{M}_{34} & \mathbf{M}_{35} & \mathbf{M}_{36} & \mathbf{M}_{37} & \mathbf{M}_{38} & \mathbf{M}_{39} & \mathbf{M}_{3,10} & \mathbf{M}_{3,11} & \mathbf{M}_{3,12}\end{bmatrix} \tag{251}$$

Then, we can solve this matrix row by row. The first row of this matrix can be denoted as:

$$\mathbf{M}_{11} = \lfloor {}^G\mathbf{p}_{fs} - {}^G\mathbf{p}_{I_k}\rfloor {}^G_{I_k}\mathbf{R}\mathbf{\Phi}_{I11} - \mathbf{\Phi}_{I51} = \lfloor {}^G\mathbf{p}_{fs} - {}^G\mathbf{p}_{I_0} - {}^G\mathbf{v}_{I_0}\delta t_k + \frac{1}{2}{}^G\mathbf{g}\delta t_k^2\rfloor {}^G_{I_0}\mathbf{R} \tag{252}$$

$$\mathbf{M}_{12} = \lfloor {}^G\mathbf{p}_{fs} - {}^G\mathbf{p}_{I_k}\rfloor {}^G_{I_k}\mathbf{R}\mathbf{\Phi}_{I12} - \mathbf{\Phi}_{I52} \tag{253}$$

$$\mathbf{M}_{13} = -\mathbf{\Phi}_{I53} = -\mathbf{I}_3\delta t_k \tag{254}$$

$$\mathbf{M}_{14} = -\mathbf{\Phi}_{I54} \tag{255}$$

$$\mathbf{M}_{15} = -\mathbf{I}_3 \tag{256}$$

$$\mathbf{M}_{16} = \mathbf{I}_3 \tag{257}$$

$$\mathbf{M}_{17} = \mathbf{M}_{19} = \mathbf{M}_{1,12} = \mathbf{0}_3 \tag{258}$$

$$\mathbf{M}_{18} = \mathbf{M}_{1,10} = \mathbf{M}_{1,11} = \mathbf{0}_{3\times 1} \tag{259}$$

The second row of this matrix can be described as:

$$\mathbf{M}_{21} = \lfloor {}^G_{T_k}\hat{\mathbf{R}}^T \hat{\mathbf{p}}_{ft} - {}^G\hat{\mathbf{p}}_{I_k} + {}^G\hat{\mathbf{p}}_{T_k} \rfloor {}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Phi}_{I11} - \mathbf{\Phi}_{I51} \tag{260}$$

$$= \lfloor {}^G_{T_k}\hat{\mathbf{R}}^T \hat{\mathbf{p}}_{ft} + {}^G\hat{\mathbf{p}}_{T_k} - {}^G\hat{\mathbf{p}}_{I_0} - {}^G\hat{\mathbf{v}}_{I_0}\delta t_k + \frac{1}{2}{}^G\mathbf{g}\delta t_k^2 \rfloor {}^G_{I_0}\hat{\mathbf{R}} \tag{261}$$

$$\mathbf{M}_{22} = \lfloor {}^G_{T_k}\hat{\mathbf{R}}^T \hat{\mathbf{p}}_{ft} - {}^G\hat{\mathbf{p}}_{I_k} + {}^G\hat{\mathbf{p}}_{T_k} \rfloor {}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Phi}_{I12} - \mathbf{\Phi}_{I52} \tag{262}$$

$$\mathbf{M}_{23} = -\mathbf{\Phi}_{I53} = -\mathbf{I}_3\delta t_k \tag{263}$$

$$\mathbf{M}_{24} = -\mathbf{\Phi}_{I54} \tag{264}$$

$$\mathbf{M}_{25} = -\mathbf{I}_3 \tag{265}$$

$$\mathbf{M}_{26} = \mathbf{0}_3 \tag{266}$$

$$\mathbf{M}_{27} = -{}^G_{T_k}\hat{\mathbf{R}}\lfloor {}^T\hat{\mathbf{p}}_{ft} \rfloor \mathbf{\Phi}_{T11} + \mathbf{\Phi}_{T31} = \lfloor -{}^G_{T_k}\hat{\mathbf{R}}^T \hat{\mathbf{p}}_{ft} - {}^G\hat{\mathbf{p}}_{T_k} + {}^G\hat{\mathbf{p}}_{T_0} \rfloor {}^G_{T_0}\hat{\mathbf{R}} \tag{267}$$

$$\mathbf{M}_{28} = -{}^G_{T_k}\hat{\mathbf{R}}\lfloor {}^T\hat{\mathbf{p}}_{ft} \rfloor \mathbf{\Phi}_{T12} + \mathbf{\Phi}_{T32} \tag{268}$$

$$\mathbf{M}_{29} = \mathbf{I}_3 \tag{269}$$

$$\mathbf{M}_{2,10} = \mathbf{\Phi}_{T34} = \int_{t_0}^{t} {}^G_{T_\tau}\mathbf{R}\mathbf{d}\tau\mathbf{e}_1 \tag{270}$$

$$\mathbf{M}_{2,11} = \mathbf{\Phi}_{T35} = \int_{t_0}^{t} {}^G_{T_\tau}\mathbf{R}\mathbf{d}\tau\mathbf{e}_2 \tag{271}$$

$$\mathbf{M}_{2,12} = {}^G_{T_k}\hat{\mathbf{R}} \tag{272}$$

The third row of this matrix can be described as:

$$\mathbf{M}_{31} = \lfloor -{}^G\hat{\mathbf{p}}_{I_k} + {}^G\hat{\mathbf{p}}_{T_k} \rfloor {}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Phi}_{I11} - \mathbf{\Phi}_{I51} = \lfloor {}^G\hat{\mathbf{p}}_{T_k} - {}^G\hat{\mathbf{p}}_{I_0} - {}^G\hat{\mathbf{v}}_{I_0}\delta t_k + \frac{1}{2}{}^G\mathbf{g}\delta t_k^2 \rfloor {}^G_{I_0}\hat{\mathbf{R}} \tag{273}$$

$$\mathbf{M}_{32} = \lfloor -{}^G\hat{\mathbf{p}}_{I_k} + {}^G\hat{\mathbf{p}}_{T_k} \rfloor {}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Phi}_{I12} - \mathbf{\Phi}_{I52} \tag{274}$$

$$\mathbf{M}_{33} = -\mathbf{\Phi}_{I53} = -\mathbf{I}_3\delta t_k \tag{275}$$

$$\mathbf{M}_{34} = -\mathbf{\Phi}_{I54} \tag{276}$$

$$\mathbf{M}_{35} = -\mathbf{I}_3 \tag{277}$$

$$\mathbf{M}_{36} = \mathbf{0}_3 \tag{278}$$

$$\mathbf{M}_{37} = \mathbf{\Phi}_{T31} = \lfloor -{}^G\hat{\mathbf{p}}_{T_k} + {}^G\hat{\mathbf{p}}_{T_0} \rfloor {}^G_{T_0}\hat{\mathbf{R}} \tag{279}$$

$$\mathbf{M}_{38} = \mathbf{\Phi}_{T32} \tag{280}$$

$$\mathbf{M}_{39} = \mathbf{I}_3 \tag{281}$$

$$\mathbf{M}_{3,10} = \mathbf{\Phi}_{T34} = \int_{t_0}^{t} {}^G_{T_\tau}\mathbf{R}\mathbf{d}\tau\mathbf{e}_1 \tag{282}$$

$$\mathbf{M}_{3,11} = \mathbf{\Phi}_{T35} = \int_{t_0}^{t} {}^G_{T_\tau}\mathbf{R}\mathbf{d}\tau\mathbf{e}_2 \tag{283}$$

$$\mathbf{M}_{3,12} = \mathbf{0}_3 \tag{284}$$

Similarly, based on the constant angular velocity ${}^T\boldsymbol{\omega}_z$ and linear velocity ${}^Tv_x$ and ${}^Tv_y$ assumption, we can have that:

$$
{}^{T_0}_{T_k}\hat{\mathbf{R}} = \begin{bmatrix} \cos(\hat{\omega}_z \delta t_k) & -\sin(\hat{\omega}_z \delta t_k) & 0 \\ \sin(\hat{\omega}_z \delta t_k) & \cos(\hat{\omega}_z \delta t_k) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{285}
$$

$$
\boldsymbol{\Phi}_{T12} = \int_{t_0}^{t} {}^{T_k}_{T_\tau}\mathbf{R} \mathbf{d}\tau \mathbf{e}_3 = \int_{t_0}^{t} \begin{bmatrix} \cos(\hat{\omega}_z(\tau - t)) & -\sin(\hat{\omega}_z(\tau - t)) & 0 \\ \sin(\hat{\omega}_z(\tau - t)) & \cos(\hat{\omega}_z(\tau - t)) & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{d}\tau \mathbf{e}_3 = \mathbf{e}_3 \delta t_k \tag{286}
$$

$$
\boldsymbol{\Phi}_{T34}\hat{\omega}_z = \int_{t_0}^{t} {}^{G}_{T_\tau}\mathbf{R} \mathbf{d}\tau \mathbf{e}_1 \hat{\omega}_z = {}^{G}_{T_0}\hat{\mathbf{R}} \int_{t_0}^{t} {}^{T_0}_{T_\tau}\mathbf{R} \mathbf{d}\tau \mathbf{e}_1 \hat{\omega}_z = {}^{G}_{T_0}\hat{\mathbf{R}} \int_{t_0}^{t} \begin{bmatrix} \cos(\hat{\omega}_z \delta\tau) \\ \sin(\hat{\omega}_z \delta\tau) \\ 0 \end{bmatrix} \mathbf{d}\tau \boldsymbol{\omega}_z \tag{287}
$$

$$
= {}^{G}_{T_0}\hat{\mathbf{R}} \left(- \begin{bmatrix} -\sin(\hat{\omega}_z \delta t_k) \\ \cos(\hat{\omega}_z \delta t_k) \\ 0 \end{bmatrix} + \mathbf{e}_2 \right) = {}^{G}_{T_0}\hat{\mathbf{R}} \left(\mathbf{I}_3 - {}^{T_0}_{T_k}\hat{\mathbf{R}} \right) \mathbf{e}_2 \tag{288}
$$

$$
\boldsymbol{\Phi}_{T35}\hat{\omega}_z = \int_{t_0}^{t} {}^{G}_{T_\tau}\mathbf{R} \mathbf{d}\tau \mathbf{e}_2 \hat{\omega}_z = {}^{G}_{T_0}\hat{\mathbf{R}} \int_{t_0}^{t} {}^{T_0}_{T_\tau}\mathbf{R} \mathbf{d}\tau \mathbf{e}_2 \boldsymbol{\omega}_z = {}^{G}_{T_0}\hat{\mathbf{R}} \int_{t_0}^{t} \begin{bmatrix} -\sin(\hat{\omega}_z \delta\tau) \\ \cos(\hat{\omega}_z \delta\tau) \\ 0 \end{bmatrix} \mathbf{d}\tau \hat{\omega}_z \tag{289}
$$

$$
= {}^{G}_{T_0}\hat{\mathbf{R}} \left(\begin{bmatrix} \cos(\hat{\omega}_z \delta t_k) \\ \sin(\hat{\omega}_z \delta t_k) \\ 0 \end{bmatrix} - \mathbf{e}_1 \right) = {}^{G}_{T_0}\hat{\mathbf{R}} \left({}^{T_0}_{T_k}\hat{\mathbf{R}} - \mathbf{I}_3 \right) \mathbf{e}_1 \tag{290}
$$

$$
\mathbf{M}_{27}\mathbf{e}_3\hat{\omega}_z = \lfloor -{}^{G}_{T_k}\hat{\mathbf{R}}^T \hat{\mathbf{p}}_{ft} - {}^{G}\hat{\mathbf{p}}_{T_k} + {}^{G}\hat{\mathbf{p}}_{T_0} \rfloor {}^{G}_{T_0}\hat{\mathbf{R}} \mathbf{e}_3 \boldsymbol{\omega}_z \tag{291}
$$

$$
= -{}^{G}_{T_k}\hat{\mathbf{R}} \lfloor {}^{T}\hat{\mathbf{p}}_{ft} \rfloor \hat{\omega}_z \mathbf{e}_3 - \lfloor {}^{G}\hat{\mathbf{p}}_{T_k} - {}^{G}\hat{\mathbf{p}}_{T_0} \rfloor {}^{G}_{T_0}\hat{\mathbf{R}} \mathbf{e}_3 \hat{\omega}_z \tag{292}
$$

$$
= -{}^{G}_{T_k}\hat{\mathbf{R}} \lfloor {}^{T}\hat{\mathbf{p}}_{ft} \rfloor \hat{\omega}_z \mathbf{e}_3 - \lfloor {}^{G}\hat{\mathbf{p}}_{T_0} + {}^{G}_{T_0}\hat{\mathbf{R}} \int_{t_0}^{t} {}^{T_0}_{T_\tau}\mathbf{R}^{T_\tau}\mathbf{v} \mathbf{d}\tau - {}^{G}\hat{\mathbf{p}}_{T_0} \rfloor {}^{G}_{T_0}\hat{\mathbf{R}} \mathbf{e}_3 \hat{\omega}_z \tag{293}
$$

$$
= -{}^{G}_{T_k}\hat{\mathbf{R}} \lfloor {}^{T}\hat{\mathbf{p}}_{ft} \rfloor \hat{\omega}_z \mathbf{e}_3 + \lfloor {}^{G}_{T_0}\hat{\mathbf{R}} \left({}^{T_0}_{T_k}\hat{\mathbf{R}} - \mathbf{I}_3 \right) \mathbf{e}_2 \rfloor {}^{G}_{T_0}\hat{\mathbf{R}} \mathbf{e}_3 {}^{T}\hat{v}_x - \lfloor {}^{G}_{T_0}\hat{\mathbf{R}} \left({}^{T_0}_{T_k}\hat{\mathbf{R}} - \mathbf{I}_3 \right) \mathbf{e}_1 \rfloor {}^{G}_{T_0}\hat{\mathbf{R}} \mathbf{e}_3 {}^{T}\hat{v}_y \tag{294}
$$

$$
= -{}^{G}_{T_k}\hat{\mathbf{R}} \lfloor {}^{T}\hat{\mathbf{p}}_{ft} \rfloor \hat{\omega}_z \mathbf{e}_3 + {}^{G}_{T_k}\hat{\mathbf{R}} \mathbf{e}_1 {}^{T}\hat{v}_x + {}^{G}_{T_k}\hat{\mathbf{R}} \mathbf{e}_2 {}^{T}\hat{v}_y - {}^{G}_{T_0}\hat{\mathbf{R}} \mathbf{e}_1 {}^{T}\hat{v}_x - {}^{G}_{T_0}\hat{\mathbf{R}} \mathbf{e}_2 {}^{T}\hat{v}_y \tag{295}
$$

Therefore, we can rewrite the equation as:

$$
\mathbf{M}_k^{(3)} = \mathbf{H}_{\mathbf{x}k} \boldsymbol{\Phi}^{(3)}(k, 0) \tag{296}
$$

$$
= \begin{bmatrix} \mathbf{H}_{C1}{}^{I}_{G}\hat{\mathbf{R}} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{H}_{C2}{}^{I}_{G}\hat{\mathbf{R}} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}_{C3}{}^{I}_{G}\hat{\mathbf{R}} \end{bmatrix} \times
$$

$$
\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & -\mathbf{I}_3\delta t_k & \mathbf{M}_{14} & -\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_{3\times1} & \mathbf{0}_3 & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_3 \\ \mathbf{M}_{21} & \mathbf{M}_{22} & -\mathbf{I}_3\delta t_k & \mathbf{M}_{24} & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{M}_{27} & \mathbf{M}_{28} & \mathbf{I}_3 & \mathbf{M}_{2,10} & \mathbf{M}_{2,11} & {}^{G}_{T_k}\hat{\mathbf{R}} \\ \mathbf{M}_{31} & \mathbf{M}_{32} & -\mathbf{I}_3\delta t_k & \mathbf{M}_{34} & -\mathbf{I}_3 & \mathbf{0}_3 & \lfloor -{}^{G}\hat{\mathbf{p}}_{T_k} + {}^{G}\hat{\mathbf{p}}_{T_0} \rfloor {}^{G}_{T_0}\hat{\mathbf{R}} & \mathbf{M}_{38} & \mathbf{I}_3 & \mathbf{M}_{3,10} & \mathbf{M}_{3,11} & \mathbf{0}_3 \end{bmatrix} \tag{297}
$$

The unobservable directions $\mathbf{N}^{(3)}$ span the right null space of the observability matrix $\mathbf{M}^{(3)}$, that is: $\mathbf{M}^{(3)}\mathbf{N}^{(3)} = \mathbf{0}$. Based on the derivation of the observability matrix, we can have the unobservable directions as:

$$\mathbf{N}^{(3)} = \begin{bmatrix} \mathbf{N}_1^{(3)} & \mathbf{N}_{2:4}^{(3)} & \mathbf{N}_{{}^T_G\mathbf{R}}^{(3)} \end{bmatrix} = \begin{bmatrix} {}^{I_0}_{G}\hat{\mathbf{R}}{}^G\mathbf{g} & \mathbf{0}_3 & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times1} & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^G\hat{\mathbf{v}}_{I_0} \rfloor {}^G\mathbf{g} & \mathbf{0}_3 & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times1} & \mathbf{0}_3 & \mathbf{0}_{3\times1} \\ -\lfloor {}^G\hat{\mathbf{p}}_{I_0} \rfloor {}^G\mathbf{g} & \mathbf{I}_3 & \mathbf{0}_{3\times1} \\ -\lfloor {}^G\hat{\mathbf{p}}_{fs} \rfloor {}^G\mathbf{g} & \mathbf{I}_3 & \mathbf{0}_{3\times1} \\ -{}^{T_0}_{I_0}\hat{\mathbf{R}}{}^G\mathbf{g} & \mathbf{0}_3 & \mathbf{e}_3 \\ \mathbf{0}_1 & \mathbf{0}_3 & \mathbf{0}_1 \\ -\lfloor {}^G\hat{\mathbf{p}}_{T_0} \rfloor {}^G\mathbf{g} & \mathbf{I}_3 & \mathbf{0}_{3\times1} \\ \mathbf{0}_1 & \mathbf{0}_{1\times3} & \hat{v}_y \\ \mathbf{0}_1 & \mathbf{0}_{1\times3} & -\hat{v}_x \\ \mathbf{0}_{3\times1} & \mathbf{0}_3 & \lfloor {}^T\hat{\mathbf{p}}_{ft} \rfloor \mathbf{e}_3 \end{bmatrix} \tag{298}$$

However, If without the measurement of the target representative point, the system will have additional 3 unobservable directions related to the target representative point position as:

$$\mathbf{N}_{{}^G\mathbf{p}_T}^{(3)} = \begin{bmatrix} \mathbf{0}_{1\times15} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_1 & \left({}^G_{T_0}\hat{\mathbf{R}}\mathbf{e}_3\right)^\top & \mathbf{0}_1 & \mathbf{0}_1 & -\mathbf{e}_3^\top \\ \mathbf{0}_{1\times15} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_1 & \left({}^G_{T_0}\hat{\mathbf{R}}\mathbf{e}_2\right)^\top & -\hat{\omega}_z & \mathbf{0}_1 & -\mathbf{e}_2^\top \\ \mathbf{0}_{1\times15} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_1 & \left({}^G_{T_0}\hat{\mathbf{R}}\mathbf{e}_1\right)^\top & \mathbf{0}_1 & \hat{\omega}_z & -\mathbf{e}_1^\top \end{bmatrix}^\top \tag{299}$$

6.4 Geometrical Interpretation - Verification of $\mathbf{N}_{{}^T_G\mathbf{R}}^{(3)}$

We assume there is a small angle $\delta\theta$ change to the orientation, then, the disturbance to the state can be written as:

$${}^{T'}_{G}\mathbf{R} = {}^{T'}_{T}\mathbf{R}{}^{T}_{G}\mathbf{R} = (\mathbf{I}_3 - \lfloor \mathbf{e}_3 \rfloor \delta\theta)\, {}^T_G\mathbf{R} \tag{300}$$

$${}^{T'}\mathbf{p}_{ft} = {}^{T'}_{T}\mathbf{R}{}^{T}\mathbf{p}_{ft} = (\mathbf{I}_3 - \lfloor \mathbf{e}_3 \rfloor \delta\theta)\, {}^T\mathbf{p}_{ft} \simeq {}^T\mathbf{p}_{ft} + \lfloor {}^T\mathbf{p}_{ft} \rfloor \mathbf{e}_3 \delta\theta \tag{301}$$

$${}^{T'}\mathbf{v}_{T'} = {}^{T'}_{T}\mathbf{R}{}^{T}\mathbf{v}_{T} = (\mathbf{I}_3 - \lfloor \mathbf{e}_3 \delta\theta \rfloor)\, {}^T\mathbf{v}_T = {}^T\mathbf{v}_T + \lfloor {}^T\mathbf{v}_T \rfloor \mathbf{e}_3 \delta\theta = {}^T\mathbf{v}_T + \begin{bmatrix} v_y \\ -v_x \\ 0 \end{bmatrix} \delta\theta \tag{302}$$

Similar to previous sections, the target feature measurements and the target representative point measurements will remain the same:

$${}^I\mathbf{p}_{ft'} = {}^I_G\mathbf{R} \left({}^G_{T'}\mathbf{R}{}^{T'}\mathbf{p}_{ft} - {}^G\mathbf{p}_I + {}^G\mathbf{p}_{T'} \right) = {}^I_G\mathbf{R} \left({}^G_T\mathbf{R}{}^T_{T'}\mathbf{R}{}^{T'}_{T}\mathbf{R}{}^T\mathbf{p}_{ft} - {}^G\mathbf{p}_I + {}^G\mathbf{p}_T \right) \tag{303}$$

$$= {}^I\mathbf{p}_{ft} \tag{304}$$

$${}^I\mathbf{p}_{T'} = {}^I_G\mathbf{R} \left({}^G\mathbf{p}_{T'} - {}^G\mathbf{p}_I \right) = {}^I_G\mathbf{R} \left({}^G\mathbf{p}_T - {}^G\mathbf{p}_I \right) \tag{305}$$

$$= {}^I\mathbf{p}_T \tag{306}$$

Hence, the disturbed error states can be written as:

$$\tilde{\mathbf{x}}' = \begin{bmatrix} \tilde{\mathbf{x}}_I' \\ {}^G\tilde{\mathbf{p}}_{fs}' \\ \delta\theta_{T'} \\ {}^{T'}\tilde{\boldsymbol{\omega}} \\ {}^G\tilde{\mathbf{p}}_{T'} \\ {}^{T'}\tilde{v}_x \\ {}^{T'}\tilde{v}_y \\ {}^{T'}\tilde{\mathbf{p}}_{ft} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{x}}_I \\ {}^G\tilde{\mathbf{p}}_{fs} \\ \delta\theta_T + \mathbf{e}_3\delta\theta \\ {}^T\tilde{\boldsymbol{\omega}} \\ {}^G\tilde{\mathbf{p}}_T \\ {}^T\tilde{v}_x + \hat{v}_y\delta\theta \\ {}^T\tilde{v}_y - \hat{v}_x\delta\theta \\ {}^T\tilde{\mathbf{p}}_{ft} + \lfloor {}^T\hat{\mathbf{p}}_{ft}\rfloor \mathbf{e}_3\delta\theta \end{bmatrix} = \tilde{\mathbf{x}} + \begin{bmatrix} \mathbf{0}_{18\times 1} \\ \mathbf{e}_3 \\ \mathbf{0}_1 \\ \mathbf{0}_{3\times 1} \\ \hat{v}_y \\ -\hat{v}_x \\ \lfloor {}^T\hat{\mathbf{p}}_{ft}\rfloor \mathbf{e}_3 \end{bmatrix} \delta\theta = \tilde{\mathbf{x}} + \mathbf{N}^{(3)}_{{}^T_G\mathbf{R}}\delta\theta \tag{307}$$

6.5 Geometrical Interpretation - Verification of $\mathbf{N}^{(3)}_{{}^G\mathbf{p}_T}$

If we disturb the target position by δp along the direction of ${}^G_{T_0}\hat{\mathbf{R}}\mathbf{e}_3$, then we will have:

$$ {}^G\mathbf{p}_{T'} = {}^G\mathbf{p}_T + {}^G_{T_0}\mathbf{R}\mathbf{e}_3\delta p \tag{308}$$
$$ {}^T\mathbf{p}_{T'} = {}^T_G\mathbf{R}\left({}^G\mathbf{p}_{T'} - {}^G\mathbf{p}_T\right) = \mathbf{e}_3\delta p \tag{309}$$
$$ {}^{T'}\mathbf{p}_{ft} = {}^{T'}_T\mathbf{R}\left({}^T\mathbf{p}_{ft} - \mathbf{e}_3\delta p\right) \tag{310}$$

Similar to previous sections, the target feature measurements will remain the same. Hence, the disturbed error state can be written as:

$$\tilde{\mathbf{x}}' = \begin{bmatrix} \tilde{\mathbf{x}}_I' \\ {}^G\tilde{\mathbf{p}}_{fs}' \\ \delta\theta_{T'} \\ {}^{T'}\tilde{\boldsymbol{\omega}} \\ {}^G\tilde{\mathbf{p}}_{T'} \\ {}^{T'}\tilde{v}_x \\ {}^{T'}\tilde{v}_y \\ {}^{T'}\tilde{\mathbf{p}}_{ft} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{x}}_I \\ {}^G\tilde{\mathbf{p}}_{fs} \\ \delta\theta_T \\ {}^T\tilde{\boldsymbol{\omega}} \\ {}^G\tilde{\mathbf{p}}_T + {}^G_{T_0}\hat{\mathbf{R}}\mathbf{e}_3\delta p \\ {}^T\tilde{v}_x \\ {}^T\tilde{v}_y \\ {}^T\tilde{\mathbf{p}}_{ft} - \mathbf{e}_3\delta p \end{bmatrix} = \tilde{\mathbf{x}} + \begin{bmatrix} \mathbf{0}_{18\times 1} \\ \mathbf{0}_{3\times 1} \\ \mathbf{0}_{3\times 1} \\ {}^G_{T_0}\hat{\mathbf{R}}\mathbf{e}_3 \\ 0 \\ 0 \\ -\mathbf{e}_3 \end{bmatrix} \delta p \tag{311}$$

Follow the same way, if we disturb the target position by δp along the direction of ${}^G_T\mathbf{R}\mathbf{e}_2$, then we will have the disturbed state as:

$$ {}^G\mathbf{p}_{T'} = {}^G\mathbf{p}_T + {}^G_T\mathbf{R}\mathbf{e}_2\delta p \tag{312}$$
$$ {}^T\mathbf{p}_{T'} = {}^T_G\mathbf{R}\left({}^G\mathbf{p}_{T'} - {}^G\mathbf{p}_T\right) = \mathbf{e}_2\delta p \tag{313}$$
$$ {}^{T'}\mathbf{p}_{ft} = {}^{T'}_T\mathbf{R}\left({}^T\mathbf{p}_{ft} - \mathbf{e}_2\delta p\right) \tag{314}$$
$$ {}^{T'}\mathbf{v}_{T'} = {}^{T'}_T\mathbf{R}\left({}^T\mathbf{v}_T + \lfloor \omega_z\mathbf{e}_3\rfloor \mathbf{e}_2\delta p\right) \tag{315}$$

Similarly, the target feature measurements will remain the same. Hence, the disturbed error state can be written as:

$$\tilde{\mathbf{x}}' = \begin{bmatrix} \tilde{\mathbf{x}}_I' \\ {}^G\tilde{\mathbf{p}}_{fs}' \\ \delta\theta_{T'} \\ {}^{T'}\tilde{\boldsymbol{\omega}} \\ {}^G\tilde{\mathbf{p}}_{T'} \\ {}^{T'}\tilde{v}_x \\ {}^{T'}\tilde{v}_y \\ {}^{T'}\tilde{\mathbf{p}}_{ft} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{x}}_I \\ {}^G\tilde{\mathbf{p}}_{fs} \\ \delta\theta_T \\ {}^T\tilde{\boldsymbol{\omega}} \\ {}^G\tilde{\mathbf{p}}_T + {}^G_T\hat{\mathbf{R}}\mathbf{e}_2\delta p \\ {}^T\tilde{v}_x - \hat{\omega}_z\delta p \\ {}^T\tilde{v}_y \\ {}^T\tilde{\mathbf{p}}_{ft} - \mathbf{e}_2\delta p \end{bmatrix} = \tilde{\mathbf{x}} + \begin{bmatrix} \mathbf{0}_{18\times 1} \\ \mathbf{0}_{3\times 1} \\ \mathbf{0}_{3\times 1} \\ {}^G_{T_0}\hat{\mathbf{R}}\mathbf{e}_2 \\ -\hat{\omega}_z \\ 0 \\ -\mathbf{e}_2 \end{bmatrix} \delta p \tag{316}$$

Again, if we disturb the target position by δp along the direction of ${}^G_T\mathbf{R}\mathbf{e}_1$, then we will have the disturbed state and error states as:

$$
\begin{aligned}
{}^G\mathbf{p}_{T'} &= {}^G\mathbf{p}_T + {}^G_T\mathbf{R}\mathbf{e}_1\delta p && (317)\\
{}^T\mathbf{p}_{T'} &= {}^T_G\mathbf{R}\left({}^G\mathbf{p}_{T'} - {}^G\mathbf{p}_T\right) = \mathbf{e}_1\delta p && (318)\\
{}^{T'}\mathbf{p}_{ft} &= {}^{T'}_T\mathbf{R}\left({}^T\mathbf{p}_{ft} - \mathbf{e}_1\delta p\right) && (319)\\
{}^{T'}\mathbf{v}_{T'} &= {}^{T'}_T\mathbf{R}\left({}^T\mathbf{v}_T + \lfloor\omega_z\mathbf{e}_3\rfloor\mathbf{e}_1\delta p\right) && (320)
\end{aligned}
$$

Similarly, the target feature measurements will remain the same. Hence, the disturbed error state can be written as:

$$
\tilde{\mathbf{x}}' = \begin{bmatrix} \tilde{\mathbf{x}}'_I \\ {}^G\tilde{\mathbf{p}}'_{fs} \\ \delta\theta_{T'} \\ {}^{T'}\tilde{\boldsymbol{\omega}} \\ {}^G\tilde{\mathbf{p}}_{T'} \\ {}^{T'}\tilde{v}_x \\ {}^{T'}\tilde{v}_y \\ {}^{T'}\tilde{\mathbf{p}}_{ft} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{x}}_I \\ {}^G\tilde{\mathbf{p}}_{fs} \\ \delta\theta_T \\ {}^T\tilde{\boldsymbol{\omega}} \\ {}^G\tilde{\mathbf{p}}_T + {}^G_T\hat{\mathbf{R}}\mathbf{e}_2\delta p \\ {}^T\tilde{v}_x \\ {}^T\tilde{v}_y + \hat{\omega}_z\delta p \\ {}^T\tilde{\mathbf{p}}_{ft} - \mathbf{e}_3\delta p \end{bmatrix} = \tilde{\mathbf{x}} + \begin{bmatrix} \mathbf{0}_{18\times 3} \\ \mathbf{0}_3 \\ \mathbf{0}_3 \\ {}^G_{T_0}\hat{\mathbf{R}}\mathbf{e}_1 \\ 0 \\ \hat{\omega}_z \\ -\mathbf{e}_1 \end{bmatrix}\delta p \tag{321}
$$

7 Summary and Discussion

Given motion model 1 (constant ${}^G\mathbf{v}_T$ and constant ${}^T\boldsymbol{\omega}$), with static feature, target feature, and representative point measurements, the system will have at least 7 unobservable directions related to global yaw, global IMU position ${}^G\mathbf{p}_I$, and the target orientation ${}^T_G\mathbf{R}$. The first 4 unobservable directions are inherited from VINS [2]. If measurements of the target's representative point are *unavailable* (due to occlusion), the system will have one more unobservable direction related to the representative point position along the rotation axis of ${}^T\boldsymbol{\omega}$. The standard and extra unobservable directions related to the target can be respectively written as:

$$
\mathbf{N}^{(1)}_{{}^T_G\mathbf{R}} = \begin{bmatrix} \mathbf{0}_{3\times 15} & \mathbf{0}_3 & \mathbf{I}_3 & \left(\lfloor{}^T\hat{\boldsymbol{\omega}}\rfloor\right)^\top & \mathbf{0}_3 & \mathbf{0}_3 & \left(\lfloor{}^T\hat{\mathbf{p}}_{ft}\rfloor\right)^\top \end{bmatrix}^\top \tag{322}
$$

$$
\mathbf{N}^{(1)}_{{}^G\hat{\mathbf{p}}_T} = \begin{bmatrix} \mathbf{0}_{1\times 15} & \mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 3} & \left({}^G_{T_0}\hat{\mathbf{R}}^T\hat{\boldsymbol{\omega}}\right)^\top & \mathbf{0}_{1\times 3} & \left(-{}^T\hat{\boldsymbol{\omega}}\right)^\top \end{bmatrix}^\top \tag{323}
$$

For motion model 2 (constant ${}^T\mathbf{v}_T$ and constant ${}^T\boldsymbol{\omega}$), with all measurements, the system will also have at least 7 unobservable directions, which have the same geometric interpretation as those of model 1. Similarly, without representative point measurements, the system will have 3 additional unobservable directions related to the full 3D position of the representative point. The standard and extra unobservable directions related to the target can be respectively written as:

$$
\mathbf{N}^{(2)}_{{}^T_G\mathbf{R}} = \begin{bmatrix} \mathbf{0}_{3\times 15} & \mathbf{0}_3 & \mathbf{I}_3 & \left(\lfloor{}^T\boldsymbol{\omega}\rfloor\right)^\top & \mathbf{0}_3 & \left(\lfloor{}^T\hat{\mathbf{v}}_T\rfloor\right)^\top & \left(\lfloor{}^T\hat{\mathbf{p}}_{ft}\rfloor\right)^\top \end{bmatrix}^\top \tag{324}
$$

$$
\mathbf{N}^{(2)}_{{}^G\mathbf{p}_T} = \begin{bmatrix} \mathbf{0}_{3\times 15} & \mathbf{0}_3 & \mathbf{0}_3 & \left({}^G_{T_0}\hat{\mathbf{R}}\right)^\top & \left(\lfloor{}^T\hat{\boldsymbol{\omega}}\rfloor\right)^\top & -\mathbf{I}_3 \end{bmatrix}^\top \tag{325}
$$

For motion model 3 (planar motion with constant ω_z, v_x, and v_y), with all measurements, unlike with the above two models, the target's roll and pitch become observable and the system has at least

5 unobservable directions where 4 are inherited from VINS and 1 is related to target orientation yaw. Without the representative point measurements, similar to model 2, the system will also gain additional 3 unobservable directions related to the full 3D position of the representative point. The standard and extra unobservable directions related to the target can be respectively written as:

$$\mathbf{N}^{(3)}_{{}^T_G\mathbf{R}} = \begin{bmatrix} \mathbf{0}_{1\times15} & \mathbf{0}_{1\times3} & \mathbf{e}_3^\top & \mathbf{0}_1 & \mathbf{0}_{3\times1} & \hat{v}_y & -\hat{v}_x & \left(\lfloor {}^T\hat{\mathbf{p}}_{ft} \rfloor \mathbf{e}_3\right)^\top \end{bmatrix}^\top \tag{326}$$

$$\mathbf{N}^{(3)}_{{}^G\mathbf{p}_T} = \begin{bmatrix} \mathbf{0}_{1\times15} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_1 & \left({}^G_{T_0}\hat{\mathbf{R}}\mathbf{e}_3\right)^\top & \mathbf{0}_1 & \mathbf{0}_1 & -\mathbf{e}_3^\top \\ \mathbf{0}_{1\times15} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_1 & \left({}^G_{T_0}\hat{\mathbf{R}}\mathbf{e}_2\right)^\top & -\omega_z & \mathbf{0}_1 & -\mathbf{e}_2^\top \\ \mathbf{0}_{1\times15} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_1 & \left({}^G_{T_0}\hat{\mathbf{R}}\mathbf{e}_1\right)^\top & \mathbf{0}_1 & \omega_z & -\mathbf{e}_1^\top \end{bmatrix}^\top \tag{327}$$

Based on these observability analysis results, it can be seen that VILTT systems will have at least 4 unobservable directions inherited from VINS which correspond to the initial global yaw and global IMU position, while additional unobservable directions depend on the chosen target motion model. In addition, it is not trivial to choose an appropriate representative point that will be frequently measured. A representative point that is occluded frequently or cannot be tracked reliably will make the system suffer from the introduction of additional unobservable directions. On the other hand, unobservable parameters can yield simpler initialization schemes as the initial value of these parameters can be freely chosen. For example, in model 1 and 2, the initial target orientation can be arbitrarily chosen due to its unobservability, while for model 3, the orientation initialization procedure needs to be carefully addressed based on measurements.

Table 1: Summary for unobservable directions of VILTT with the 3 proposed motion models. The number before the unobservable direction indicates its dimension.

Motion Model	**All Measurements**	**Without Representative Point**
Model 1 Constant ${}^G\mathbf{v}$ Constant ${}^T\boldsymbol{\omega}$	(1) Global yaw (3) Global IMU position (3) Target orientation ${}^T_G\mathbf{R}$ ⇒ 7 in total	(1) Global yaw (3) Global IMU sensor position (3) Target orientation ${}^T_G\mathbf{R}$ (1) Representative point position along rotation axis ⇒ 8 in total
Model 2 Constant ${}^T\mathbf{v}$ Constant ${}^T\boldsymbol{\omega}$	(1) Global yaw (3) Global IMU position (3) Target orientation ${}^T_G\mathbf{R}$ ⇒ 7 in total	(1) Global yaw (3) Global IMU position (3) Target orientation ${}^T_G\mathbf{R}$ (3) Representative point position ⇒ 10 in total
Model 3 Planar Motion Constant v_x, v_y Constant ω_z	(1) Global yaw (3) Global IMU position (1) Target orientation yaw ⇒ 5 in total	(1) Global yaw (3) Global IMU position (1) Target orientation yaw (3) Representative point position ⇒ 8 in total

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