Research Notes and Jacobians: Continuous-Time Visual-Inertial Trajectory Estimation with Event Cameras

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1 Problem Statement

Continuous-Time Visual-Inertial Trajectory Estimation with Event Cameras by Elias Mueggler et al. [4] details the combination of a spline based trajectory estimation with visual-inertial measurements through an IMU and an event based camera. This continuous-time based system is appealing since it allows the incorporation of asynchronous measurements and high frequency measurements without increasing the overall complexity of the system.

2 B-Splines Representation Background

2.1 Standard Basis Representation

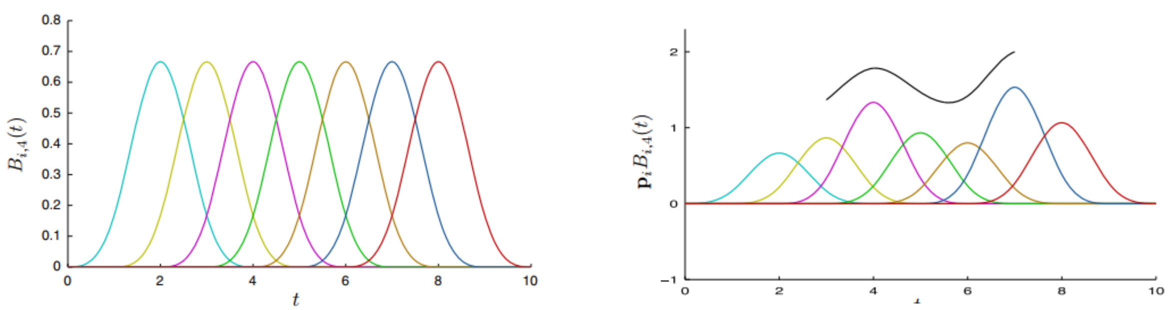

Figure 1: Seen in the left figure, we have the standard basis functions. One can see that the "contribution" of each basis function is larger in given areas along the curve. The right figure shows the combination of the control points multiplied times the basis functions seen on the left. This creates the combined continuous curve seen in back. Images taken from [5].

A B-spline is a series of control points (otherwise known as "knots") that define a continuous function that is a linear combination of the "basis functions" which are defined over the control points. We can define, for example, a linear interpolation between a 3D position as the following:

$$\mathbf{p}(t) = \sum_{i=0}^{n} \mathbf{p}_i \; B_{i,k}(t) \tag{1}$$

$$B_{i,0}(t) := \begin{cases} 1 & \text{if} \quad t_i \leq t < t_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$B_{i,k}(t) := \frac{t - t_i}{t_{i+k} - t_i} B_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} B_{i+1,k-1}(t) \tag{3}$$

where $B_{i,k}(t)$ is the De Boor-Cox recursive formula [2] the computes the fractional coefficient for each given control point, k is spline degree, and n is the number of control points. It can be seen that the the position at a given point t along the spline is the linear combination of each control point $\mathbf{p}_i$. The fractional amount that each control point contributes can be calculated using the recursive De Boor-Cox formula.

2.2 Uniform Standard Basis Matrix Representation

We can now look at a special case of B-Splines. To simplify the problem, we can look at a uniform distribution between control points along the B-spline. A uniform distribution can be described as if the variable t describes the distance to any point along the B-Spline and each control point is separated by some time Δt, then we can define the following time fraction:

$$u(t) = \frac{t - t_i}{t_{i+1} - t_i} = \frac{t - t_i}{\Delta t},\ u \in [0, 1) \tag{4}$$

where t_i and t_{i+1} is the first and second control pose position along the spline (can also be denoted as $t_i = i\Delta t$ if uniform time between each control pose), respectively. Having defined this $u(t)$, we can look at converting the De Boor-Cox formula (3) into a matrix representation. Kaihuai Qin [6] derived this for the standard form as the following (note that this is for uniform B-splines):

$$\mathbf{p}(t) = \mathbf{B}(u(t)) \begin{bmatrix} \mathbf{p}_0^\top \\ \vdots \\ \mathbf{p}_i^\top \\ \vdots \\ \mathbf{p}_n^\top \end{bmatrix} = \begin{bmatrix} 1 & \cdots & u(t)^i & \cdots & u(t)^{k-1} \end{bmatrix} \mathbf{M}^k \begin{bmatrix} \mathbf{p}_0^\top \\ \vdots \\ \mathbf{p}_i^\top \\ \vdots \\ \mathbf{p}_n^\top \end{bmatrix} \tag{5}$$

$$\mathbf{M}^k = \begin{bmatrix} m_{0,0} & m_{0,1} & \cdots & m_{0,n} \\ m_{1,0} & m_{1,1} & \cdots & m_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{k-1,0} & m_{k-1,1} & \cdots & m_{k-1,n} \end{bmatrix}_{k \times n} \tag{6}$$

$$m_{i,j} = \frac{1}{(k-1)!}\, \mathbf{C}_{k-1}^{k-1-i} \sum_{s=j}^{k-1} \left[(-1)^{s-j}\ \mathbf{C}_k^{s-j} (k-s-1)^{k-1-i} \right] \tag{7}$$

$$\mathbf{C}_n^i = \frac{n!}{i!(n-i)!} \tag{8}$$

This derivation can be seen in [6] where it is also derived for non-uniform B-spline curves. Thus, we can use the above matrix formulation to instantly get the the interpolated position at any place along the continuous time B-spline curve.

2.3 Cumulative Basis Representation

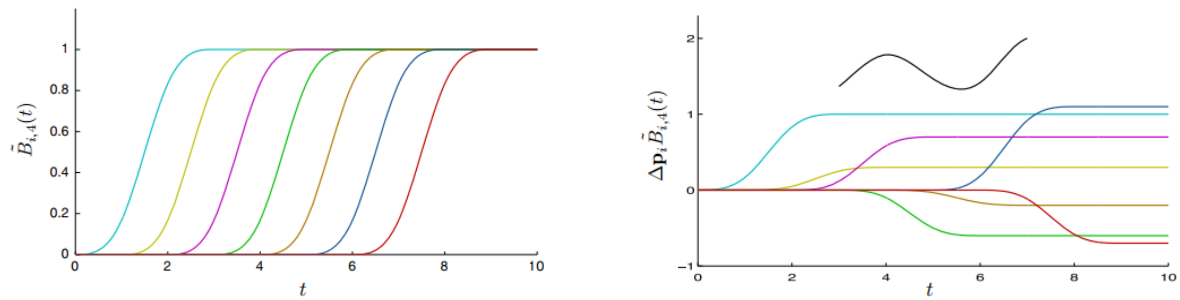

Figure 2: Seen in the left figure, we have the standard basis functions. One can see that the "contribution" of each basis function has a transition between zero to one, meaning that above a certain point the basis function will always contribute. The right figure shows the combination of the control point relatives multiplied times the basis functions seen on the left. This creates the combined continuous curve seen in back. Images taken from [5].

Now, one of the key issues with the standard representation (section 2.1) is that this causes discontinuities or do not preserve continuity properties (look into SLERP and SQUAD). It is instead better to parameters the trajectory using "cumulative" basis functions. In the case of *cubic* B-splines this ensures C^2-continuity while also preventing singularity issues with rotations. In the case of the Lie Algebra group $\mathbb{SE}(3)$, cumulative basis functions are applied locally and become free of singularities. Following Kim et al. [3], we can transform our standard basis formulation into the following cumulative basis equation:

$$\mathbf{p}(t) = \mathbf{p}_0\ \tilde{B}_{0,k}(t) + \sum_{i=1}^{n}\ (\mathbf{p}_i - \mathbf{p}_{i-1})\ \tilde{B}_{i,k}(t) \tag{9}$$

$$\tilde{B}_{i,k}(t) = \sum_{s=i}^{n} \Big[B_{s,k}(t) \Big] \tag{10}$$

where $B_{s,k}(t)$ is still defined by the De Boor-Cox recursive formulation (see equation (3)), k is spline degree, and n is the number of control points.

2.4 Uniform Cumulative Basis Matrix Representation

We can now look to create a matrix representation of this cumulative basis function. If we have a uniform distributed B-Spline (each control point is separated by some time Δt), then we can define the following time fraction:

$$u(t) = \frac{t - t_i}{t_{i+1} - t_i} = \frac{t - t_i}{\Delta t},\ \ u \in [0, 1) \tag{11}$$

Kaihuai Qin [6] derived with a summation added to take into account the cumulative nature of the basis functions:

$$\mathbf{p}(t) = \tilde{\mathbf{B}}(u(t)) \begin{bmatrix} \mathbf{p}_0^\top \\ \vdots \\ \mathbf{p}_i^\top \\ \vdots \\ \mathbf{p}_n^\top \end{bmatrix} = \begin{bmatrix} 1 & \cdots & u(t)^i & \cdots & u(t)^{k-1} \end{bmatrix} \tilde{\mathbf{M}}^k \begin{bmatrix} \mathbf{p}_0^\top \\ \vdots \\ \mathbf{p}_i^\top \\ \vdots \\ \mathbf{p}_n^\top \end{bmatrix} \tag{12}$$

$$
\tilde{\mathbf{M}}^k = \begin{bmatrix} \sum_{s=0}^{n-1} m_{0,s} & \sum_{s=1}^{n-1} m_{0,s} & \cdots & \sum_{s=n-1}^{n-1} m_{0,s} \\ \sum_{s=0}^{n-1} m_{1,s} & \sum_{s=1}^{n-1} m_{1,s} & \cdots & \sum_{s=n-1}^{n-1} m_{1,s} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{s=0}^{n-1} m_{k-1,s} & \sum_{s=1}^{n-1} m_{k-1,s} & \cdots & \sum_{s=n-1}^{n-1} m_{k-1,s} \end{bmatrix}_{k\times n} \tag{13}
$$

This can be described as summing the basis functions for all basis greater than or equal to the current (matching equation (10) exactly). This matrix can now be used to compute any cumulative B-spline of order k and with n control points.

3 Uniform Cubic B-Splines in $\mathbb{SE}(3)$

As described by Patron-Perez et al. [5], in SLAM systems we are looking for certain properties that are advantageous to the localization system and in its flexibility during sensor fusion. They define the following key characteristics that are desirable:

1. Local control, allowing the system to function online as well as in batch
2. C^2-continuity, to enable inertial predictions
3. Good approximation of minimal torque trajectories
4. A parameterization of rigid-body motion devoid of singularities

A *cubic* B-spline fits this criteria if the cumulative basis representation is used. The "local" nature of the cumulative representation prevents singularity issues with rotations that are required to be projected onto a tangent plane in respect to the $\mathbb{SO}(3)$ group. We can define the following transformation:

$$
{}_a^b\mathbf{T} = \begin{bmatrix} {}_a^b\mathbf{R} & {}^b\mathbf{p}_a \\ \mathbf{0} & 1 \end{bmatrix}, \quad {}_a^b\mathbf{T} \in \mathbb{SE}(3), \quad {}_a^b\mathbf{R} \in \mathbb{SO}(3) \tag{14}
$$

Next we assume that the angular and linear velocity are constant during each of our transformations. Using the matrix logarithm and exponential, we can define the following function:

$$
{}_s^w\mathbf{T}(t) = \exp\Big(\tilde{B}_{0,k}(t)\ \log\big({}_0^w\mathbf{T}\big)\Big) \prod_{i=1}^{n} \exp\Big(\tilde{B}_{i,k}(t)\ \log\big({}_{i-1}^{w}\mathbf{T}^{-1}\ {}_i^w\mathbf{T}\big)\Big) \tag{15}
$$

where ${}_s^w\mathbf{T}(t)$ is the position along the B-spline at time t, ${}_i^w\mathbf{T}$ are control poses, and $\tilde{B}_{i,k}(t)$ is the cumulative basis functions (see equation (10)). From here we can convert into a cubic B-spline by specifying that our $k = 4$. Note that we are a cubic spline, but our degree is given by $degree = k-1$, representing that there are at max, four control points / knots that contribute to any point along the spline curve.

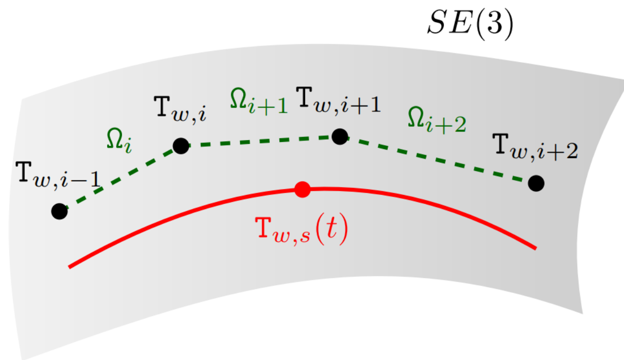

Figure 3: Pictorial representation of the control points, relative measurements, and continuous spline with the desired point shown. Image taken from [4].

As seen above, when using a cubic B-spline we only need the four transformations that are local to the position along the spline. We have a single global transformation that is then compounded three times to define our four control poses. Using this representation we can rewrite equation (15) as the following function of these four poses:

$$ {}^{w}_{s}\mathbf{T}(t) = \exp\Big(\tilde{B}_{0,4}(t)\,\log\big({}^{w}_{i-1}\mathbf{T}\big)\Big)\prod_{i=1}^{3}\exp\Big(\tilde{B}_{i,4}(t)\,\log\big({}^{w}_{i-1}\mathbf{T}^{-1}\;{}^{w}_{i}\mathbf{T}\big)\Big) \tag{16} $$

and if expanded:

$$ \begin{aligned} {}^{w}_{s}\mathbf{T}(t) = \exp\Big(\tilde{B}_{0,4}(t)\,\log\big({}^{w}_{i-1}\mathbf{T}\big)\Big)\exp\Big(\tilde{B}_{1,4}(t)\,\log\big({}^{w}_{i-1}\mathbf{T}^{-1}\;{}^{w}_{i}\mathbf{T}\big)\Big) \\ \exp\Big(\tilde{B}_{2,4}(t)\,\log\big({}^{w}_{i}\mathbf{T}^{-1}\;{}^{w}_{i+1}\mathbf{T}\big)\Big)\exp\Big(\tilde{B}_{3,4}(t)\,\log\big({}^{w}_{i+1}\mathbf{T}^{-1}\;{}^{w}_{i+2}\mathbf{T}\big)\Big) \end{aligned} \tag{17} $$

Now, to define what our basis are, we can look back at the matrix representation for the cumulative representation. Since we know $k = 4$ we can solve it to get the following matrix:

$$ \tilde{\mathbf{B}}(u(t)) = \begin{bmatrix} 1 & \cdots & u(t)^{i} & \cdots & u(t)^{4-1} \end{bmatrix} \tilde{\mathbf{M}}^{4} \tag{18} $$

$$ = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \begin{bmatrix} \sum_{s=0}^{4-1} m_{0,s} & \sum_{s=1}^{4-1} m_{0,s} & \sum_{s=2}^{4-1} m_{0,s} & \sum_{s=3}^{4-1} m_{0,s} \\ \sum_{s=0}^{4-1} m_{1,s} & \sum_{s=1}^{4-1} m_{1,s} & \sum_{s=2}^{4-1} m_{1,s} & \sum_{s=3}^{4-1} m_{1,s} \\ \sum_{s=0}^{4-1} m_{2,s} & \sum_{s=1}^{4-1} m_{2,s} & \sum_{s=2}^{4-1} m_{2,s} & \sum_{s=3}^{4-1} m_{2,s} \\ \sum_{s=0}^{4-1} m_{3,s} & \sum_{s=1}^{4-1} m_{3,s} & \sum_{s=2}^{4-1} m_{3,s} & \sum_{s=3}^{4-1} m_{3,s} \end{bmatrix}_{4\times 4} \tag{19} $$

$$ = \frac{1}{3!}\begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \begin{bmatrix} 6 & 5 & 1 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}_{4\times 4} \tag{20} $$

$$ = \frac{1}{3!}\begin{bmatrix} 6 & (5 + 3u - 3u^2 + u^3) & (1 + 3u + 3u^2 - 2u^3) & u^3 \end{bmatrix}_{1\times 4} \tag{21} $$

Note that the above matrix is the transpose of that defined in [5] as the $u(t)$ matrix is multiplied on the left instead of the right. From this, we can now write the following full form of equation (17) as the following:

$$\begin{aligned} {}^{w}_{s}\mathbf{T}(u(t)) &= \exp\Big(\frac{1}{6}(6)\,\log\big({}^{w}_{i-1}\mathbf{T}\big)\Big)\exp\Big(\frac{1}{6}(5+3u-3u^2+u^3)\,\log\big({}^{w}_{i-1}\mathbf{T}^{-1}\;{}^{w}_{i}\mathbf{T}\big)\Big) \\ &\qquad \exp\Big(\frac{1}{6}(1+3u+3u^2-2u^3)\,\log\big({}^{w}_{i}\mathbf{T}^{-1}\;{}^{w}_{i+1}\mathbf{T}\big)\Big)\exp\Big(\frac{1}{6}u^3\,\log\big({}^{w}_{i+1}\mathbf{T}^{-1}\;{}^{w}_{i+2}\mathbf{T}\big)\Big) \end{aligned} \tag{22}$$

$$\begin{aligned} &= \exp\Big(\log\big({}^{w}_{i-1}\mathbf{T}\big)\Big)\exp\Big(\frac{1}{6}(5+3u-3u^2+u^3)\,\log\big({}^{i-1}_{i}\mathbf{T}\big)\Big) \\ &\qquad \exp\Big(\frac{1}{6}(1+3u+3u^2-2u^3)\,\log\big({}^{i}_{i+1}\mathbf{T}\big)\Big)\exp\Big(\frac{1}{6}u^3\,\log\big({}^{i+1}_{i+2}\mathbf{T}\big)\Big) \end{aligned} \tag{23}$$

where $u(t) = (t - t_i)/\Delta t$ and $t_i = i\Delta t$. Note that the above equation is for a uniform time Δt between control points. Equation (23) can be used to solve for the position at any time t along the continuous cubic B-spline. It requires the two poses before the time and two after (see figure 3). If the control points are known, the position at any arbitrary time can be found.

4 Uniform Cubic B-Spline with Inertial Measurements

4.1 Inertial Measurement Model

To incorporate inertial measurements from a IMU sensor, we can leverage the continuous nature and C^2-continuity of our cubic B-spline. We can define the sensor measurement from a IMU as follows:

$$\boldsymbol{\omega}_m = \boldsymbol{\omega} + \mathbf{b}_\omega + \mathbf{n}_\omega \tag{24}$$

$$\mathbf{a}_m = \mathbf{a} + \mathbf{g} + \mathbf{b}_a + \mathbf{n}_a \tag{25}$$

where $\mathbf{g}$ is the gravity in the local frame whose global counterpart is constant (e.g., ${}^{G}\mathbf{g} \approx [0\;0\;9.81]^\top$). Now since we have cubic B-spline we can directly get the angular velocity and linear acceleration at any time t on the curve. We can re-write the above equations to take this into account as follows:

$$\boldsymbol{\omega}_m(u(t)) = \Big({}^{G}_{I}\mathbf{R}(u(t))^\top\;{}^{G}_{I}\dot{\mathbf{R}}(u(t))\Big)^\vee + \mathbf{b}_\omega + \mathbf{n}_\omega \tag{26}$$

$$\mathbf{a}_m(u(t)) = {}^{G}_{I}\mathbf{R}(u(t))^\top\Big({}^{G}\ddot{\mathbf{p}}_I(u(t)) + {}^{G}\mathbf{g}\Big) + \mathbf{b}_a + \mathbf{n}_a \tag{27}$$

where ${}^{G}_{I}\dot{\mathbf{R}}(u(t))$ and ${}^{G}\ddot{\mathbf{p}}_I(u(t))$ are sub-matrices of the derivatives of equation (23) (first and second respectively) and the $(\cdot)^\vee$ is the "vee map" operator that takes a $\lfloor\boldsymbol{\omega}(u(t))\times\rfloor$ skewsymetric matrix and maps it into the $\mathbb{R}^3$ domain. The above equations can be described as taking the global angular change and linear acceleration at time t and warping it into the IMU frame of reference. Note that we assume the B-spline is the IMU frame otherwise we would need an additional transform to warp from the sensor frame into the IMU frame of reference.

4.2 State Time Derivative

From here, we can look at taking the derivative of our state in respect to time to get these values. Our state is made of control points, so we look at taking the derivative of equation (23) in respect to time. First we can simplify equation (23) as follows:

$$ {}^{w}_{s}\mathbf{T}(u(t)) = {}^{w}_{i-1}\mathbf{T}\ \exp\Big(B_0(u(t))\ {}^{i-1}_{i}\mathbf{\Omega}\Big)\exp\Big(B_1(u(t))\ {}^{i}_{i+1}\mathbf{\Omega}\Big)\exp\Big(B_2(u(t))\ {}^{i+1}_{i+2}\mathbf{\Omega}\Big) \tag{28} $$

$$ = {}^{w}_{i-1}\mathbf{T}\ \mathbf{A}_0\ \mathbf{A}_1\ \mathbf{A}_2 \tag{29} $$

We can apply the product derivative rule to get the following:

$$ \frac{\partial}{\partial t}{}^{w}_{s}\mathbf{T}(u(t)) = {}^{w}_{i-1}\mathbf{T}\Big(\dot{\mathbf{A}}_0\ \mathbf{A}_1\ \mathbf{A}_2 + \mathbf{A}_0\ \dot{\mathbf{A}}_1\ \mathbf{A}_2 + \mathbf{A}_0\ \mathbf{A}_1\ \dot{\mathbf{A}}_2\Big) \tag{30} $$

$$ \frac{\partial}{\partial t^2}{}^{w}_{s}\mathbf{T}(u(t)) = {}^{w}_{i-1}\mathbf{T}\Big(\ddot{\mathbf{A}}_0\ \mathbf{A}_1\ \mathbf{A}_2 + \mathbf{A}_0\ \ddot{\mathbf{A}}_1\ \mathbf{A}_2 + \mathbf{A}_0\ \mathbf{A}_1\ \ddot{\mathbf{A}}_2 + 2\dot{\mathbf{A}}_0\dot{\mathbf{A}}_1\mathbf{A}_2 + 2\mathbf{A}_0\dot{\mathbf{A}}_1\dot{\mathbf{A}}_2 + 2\dot{\mathbf{A}}_0\mathbf{A}_1\dot{\mathbf{A}}_2\Big) \tag{31} $$

To find the derivatives of our matrix exponential we can look at the series definition and take the derivative from there (credit goes to Kevin Eckenhoff on how to take theses time derivatives). For example, for the first matrix exponential we can write the following:

$$ \mathbf{A}_0 = \mathbf{I} + B_0(u(t))\ {}^{i-1}_{i}\mathbf{\Omega}^\wedge + \frac{1}{2}B_0(u(t))^2\ \left({}^{i-1}_{i}\mathbf{\Omega}^\wedge\right)^2 + \frac{1}{6}B_0(u(t))^3\ \left({}^{i-1}_{i}\mathbf{\Omega}^\wedge\right)^3 + \cdots \tag{32} $$

Taking the derivative in respect to time, we can get the following:

$$ \dot{\mathbf{A}}_0 = \dot{B}_0(u(t))\ {}^{i-1}_{i}\mathbf{\Omega}^\wedge + \dot{B}_0(u(t))B_0(u(t))\ \left({}^{i-1}_{i}\mathbf{\Omega}^\wedge\right)^2 + \frac{1}{2}\dot{B}_0(u(t))B_0(u(t))^2\ \left({}^{i-1}_{i}\mathbf{\Omega}^\wedge\right)^3 + \cdots \tag{33} $$

$$ \dot{\mathbf{A}}_0 = \dot{B}_0(u(t))\ {}^{i-1}_{i}\mathbf{\Omega}^\wedge\Big(\mathbf{I} + B_0(u(t))\ \left({}^{i-1}_{i}\mathbf{\Omega}^\wedge\right) + \frac{1}{2}\dot{B}_0(u(t))B_0(u(t))^2\ \left({}^{i-1}_{i}\mathbf{\Omega}^\wedge\right)^2 + \cdots\Big) \tag{34} $$

$$ \dot{\mathbf{A}}_0 = \dot{B}_0(u(t))\ {}^{i-1}_{i}\mathbf{\Omega}^\wedge\ \mathbf{A}_0 \tag{35} $$

We can see that this is true for all $\mathbf{A}_i$ because the Ω's control point values do not vary with time. From the above equation it is simple to apply the product rule to find the second derivative as follows:

$$ \ddot{\mathbf{A}}_0 = \dot{B}_0(u(t))\ {}^{i-1}_{i}\mathbf{\Omega}^\wedge\ \dot{\mathbf{A}}_0 + \ddot{B}_0(u(t))\ {}^{i-1}_{i}\mathbf{\Omega}^\wedge\ \mathbf{A}_0 \tag{36} $$

The final piece is to find the derivative of our basis function. We can take the derivative in respect to our variable u and then chain rule this with the derivative of u in respect to time. We can perform that on the matrix form as follows:

$$ \tilde{\mathbf{B}}(u(t)) = \frac{1}{3!}\begin{bmatrix}6 & (5+3u-3u^2+u^3) & (1+3u+3u^2-2u^3) & u^3\end{bmatrix}_{1\times4} \tag{37} $$

$$ \dot{\mathbf{B}}(u(t)) = \frac{1}{3!}\begin{bmatrix}0 & (3-6u+3u^2) & (3+6u-6u^2) & 3u^2\end{bmatrix}_{1\times4} \times \frac{1}{\Delta t} \tag{38} $$

$$ \ddot{\mathbf{B}}(u(t)) = \frac{1}{3!}\begin{bmatrix}0 & (-6+6u) & (6-12u) & 6u\end{bmatrix}_{1\times4} \times \frac{1}{\Delta t^2} \tag{39} $$

This gives us the following final equation set:

$$
\begin{aligned}
{}^{w}_{s}\mathbf{T}(u(t)) &= {}^{w}_{i-1}\mathbf{T}\ \mathbf{A}_0\ \mathbf{A}_1\ \mathbf{A}_2 && (40)\\
{}^{w}_{s}\dot{\mathbf{T}}(u(t)) &= {}^{w}_{i-1}\mathbf{T}\Big(\dot{\mathbf{A}}_0\ \mathbf{A}_1\ \mathbf{A}_2 + \mathbf{A}_0\ \dot{\mathbf{A}}_1\ \mathbf{A}_2 + \mathbf{A}_0\ \mathbf{A}_1\ \dot{\mathbf{A}}_2\Big) && (41)\\
{}^{w}_{s}\ddot{\mathbf{T}}(u(t)) &= {}^{w}_{i-1}\mathbf{T}\Big(\ddot{\mathbf{A}}_0\ \mathbf{A}_1\ \mathbf{A}_2 + \mathbf{A}_0\ \ddot{\mathbf{A}}_1\ \mathbf{A}_2 + \mathbf{A}_0\ \mathbf{A}_1\ \ddot{\mathbf{A}}_2 \\
&\qquad + 2\dot{\mathbf{A}}_0\dot{\mathbf{A}}_1\mathbf{A}_2 + 2\mathbf{A}_0\dot{\mathbf{A}}_1\dot{\mathbf{A}}_2 + 2\dot{\mathbf{A}}_0\mathbf{A}_1\dot{\mathbf{A}}_2\Big) && (42)
\end{aligned}
$$

$$
\begin{aligned}
{}^{i-1}_{i}\boldsymbol{\Omega} &= \log\big({}^{w}_{i-1}\mathbf{T}^{-1}\ {}^{w}_{i}\mathbf{T}\big) && (43)\\
\mathbf{A}_j &= \exp\Big(B_j(u(t))\ {}^{i-1+j}_{i+j}\boldsymbol{\Omega}\Big) && (44)\\
\dot{\mathbf{A}}_j &= \dot{B}_j(u(t))\ {}^{i-1+j}_{i+j}\boldsymbol{\Omega}^{\wedge}\ \mathbf{A}_j && (45)\\
\ddot{\mathbf{A}}_j &= \dot{B}_j(u(t))\ {}^{i-1+j}_{i+j}\boldsymbol{\Omega}^{\wedge}\ \dot{\mathbf{A}}_j + \ddot{B}_j(u(t))\ {}^{i-1+j}_{i+j}\boldsymbol{\Omega}^{\wedge}\ \mathbf{A}_j && (46)
\end{aligned}
$$

$$
\begin{aligned}
B_0(u(t)) &= \frac{1}{3!}\ (5 + 3u - 3u^2 + u^3) && (47)\\
B_1(u(t)) &= \frac{1}{3!}\ (1 + 3u + 3u^2 - 2u^3) && (48)\\
B_2(u(t)) &= \frac{1}{3!}\ (u^3) && (49)
\end{aligned}
$$

$$
\begin{aligned}
\dot{B}_0(u(t)) &= \frac{1}{3!}\ \frac{1}{\Delta t}\ (3 - 6u + 3u^2) && (50)\\
\dot{B}_1(u(t)) &= \frac{1}{3!}\ \frac{1}{\Delta t}\ (3 + 6u - 6u^2) && (51)\\
\dot{B}_2(u(t)) &= \frac{1}{3!}\ \frac{1}{\Delta t}\ (3u^2) && (52)
\end{aligned}
$$

$$
\begin{aligned}
\ddot{B}_0(u(t)) &= \frac{1}{3!}\ \frac{1}{\Delta t^2}\ (-6 + 6u) && (53)\\
\ddot{B}_1(u(t)) &= \frac{1}{3!}\ \frac{1}{\Delta t^2}\ (6 - 12u) && (54)\\
\ddot{B}_2(u(t)) &= \frac{1}{3!}\ \frac{1}{\Delta t^2}\ (6u) && (55)
\end{aligned}
$$

4.3 Jacobians and Comments

Using the above formulation it is trivial to compute ${}^{w}_{s}\dot{\mathbf{T}}(u(t))$ and ${}^{w}_{s}\ddot{\mathbf{T}}(u(t))$ that both can be used to calculate the angular velocity and linear acceleration at a given time t. This can then be used as either a factor in a graph based method, or as a state update in a EKF framework. It is important to note that no authors to date have published a derivation of the Jacobians needed to do this update (both in [5] and [4]). They use Ceres [1] numerical differentiation to optimize, causing a large computational burden. These Jacobians are extremely complex since the above equations need to be "peurtubed" to find the Jacobian relating the acceleration to the state control points.

Appendix A: De-Boor Cox Uniform Matrix MATLAB Script

```
%% NOTES: WHAT THIS IS BASED ON
% Formula is equation 8
% K. Quin "General matrix representation for B-splines" published in 2000
% We want a size 4 since we use 4 control points
% We are using uniform matrix size, thus can use the simplified version
%% MAKE UNIFORM BSPLINE MATRIX FORM
% What size of the matrix we want (M^4)
% We have 4 control points, so k=4 (cubic spline)
k = 4;
% The prefix to all matrix entries
prefix = 1./factorial(k-1);
% Make the matrix that is k by k
matrix_abs = zeros(k,k);
% Loop through and make our matrix
for ii=0:k-1
    for jj=0:k-1
        matrix_abs(ii+1,jj+1) = m(k,ii,jj);
    end
end
% Transpose since the paper is B = M*u^t and not B = u*M
% "A Spline-Based Trajectory Representation for Sensor Fusion and Rolling
% Shutter Cameras" by Patron-Perez, Lovergrove, Sibley 2015
matrix_abs = matrix_abs';
% Cumulative, so sum the following rows
% This is because the basis functions are stacked on top of each other
% B_cum(t) = sum(B_abs, from current to last)
matrix_cum = zeros(k,k);
% Loop through and make our matrix
for ii=1:k
```

```
    % summ each row
    for extra=0:k-ii
        matrix_cum(ii,:) = matrix_cum(ii,:) + matrix_abs(ii+extra,:);
    end
end
% Print out
prefix
matrix_abs
matrix_cum
%% FUNCTIONS
% Define our C function
function val = C(ii,n)
   val = factorial(n)./(factorial(ii).*factorial(n-ii));
end
% Define our m_(i,j) function
function val = m(k,ii,jj)
   syms s
    val = C(k-1-ii,k-1).*symsum((-1).^(s-jj).*C(s-jj,k).*(k-s-1).^(k-1-ii), s, jj, k-1);
end
```

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