Section 0.1: Compound Statements (using "truth table" from Sec. 1.1)

Definition. A **mathematical statement (proposition)** is an assertive sentence that is either true or false, but not both.

Exercise 1. Which of the following is a statement? If it is a statement, determine its truth value.

(a) "Where is Ewing Hall? "

(b) "The product of an integer and an even integer is an even integer. "

(c) "Suppose this statement is false. "

(d) "$\sqrt{4+9} = \sqrt{4} + \sqrt{9}$. "

A **compound statement** is a statement form by joining one or more statements with symbols—for example, $\wedge$ "and", $\vee$ "or", $\neg$ "not".

Let p and q be two statements.

"And" (conjunction)
Written as:
Read as:

p	q	$p \wedge q$
T	T	
T	F	
F	T	
F	F	

Example 2.

- To graduate from my degree, I need to pass MATH242 and MATH210.
- $5 > 10$ and $2 < 5$. (TRUE or FALSE?)

"Or" (disjunction)
Written as:
Read as:

p	q	$p \vee q$
T	T	
T	F	
F	T	
F	F	

Example 3.

- I will eat a salad or a sandwich for lunch.
- $2 + 3 = 23$ or $3 + 4 = 7$. (TRUE or FALSE?)

> **Implication**
> Written as:
> Read as:

p	q	$p \to q$
T	T	
T	F	
F	T	
F	F	

Example 4.

- If $a = 0$, then $ab = 0$.
- "If I live in Newark, DE, then I live in the USA. "
- "If I live in Newark, DE, then I live in Canada. "
- Parents: "If it's sunny tomorrow, you may go swimming. "
- "I will give you \$ 1 million when pigs fly. " (vacuously true statement)
- Another way to remember: "If I'm drinking a cup of wine, then I must be at least 21 years old. " (T-lawful, F-unlawful)

> The **converse** of an implication $p \to q$ is the implication $q \to p$.

> **Double implication**
> Written as:
> Read as:

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$ $(p \to q) \wedge (q \to p)$
T	T			
T	F			
F	T			
F	F			

Example 5.

- I play soccer if and only if today is Friday.
- A two digits number is even if and only if it ends with 0. (TRUE or FALSE?)

Example 6. In Calculus 1, we have learned this Theorem:
$p \to q$: "A differentiable function is continuous. "

(a) What is p?

(b) What is q?

(c) What is the converse of this statement?

(d) We know that this implication is always true (thus a "Theorem"). Is the converse always true?

⋆ Suggested exercises from the textbook: Section 0.1 # 2, 3, 4 (some are assigned in Homework 1).

Summary - fill in the blanks. Answers will be posted on Canvas.

"p and q":	TRUE if	
	FALSE if	
"p or q":	TRUE if	
	FALSE if	
"p implies q":	TRUE if	
	FALSE if	
"p if and only if q":	TRUE if	
	FALSE if	

- The converse of a TRUE Theorem (implication) is not always true. When both directions are true, we have $\boxed{}$.

Sec. 0.1: Compound Statements (cont.) (using "truth table" from Sec. 1.1)

Quantifiers

- "**There exists**" (Existensial Quantifier)
  Written as:
  Read as:
- "**For all**" (Universal Quantifier)
  Written as:
  Read as:

Example 7.

- There exists a real number x that is larger than its square.
  In symbols:
- For all real numbers a and b, $ab = ba$.
  In symbols:

$\star$ **Order of quantifiers matters!**

Example 8. Consider the statement: "There is a largest real number. "
In symbols:

$\star$ What if we exchange "for all" and "there is"?

Negation ("not")
Written as:
Read as:

The **contrapositive** of the implication "$p \to q$" is the implication "$(\neg q) \to (\neg p)$".

Example 9. $p \to q$: "If I live in Newark, DE, then I live in the USA. "

- Converse:
- Contrapositive:

$\star$ **How to negate a compound statement?** First, let's see how to negate each symbol.

Negation of "and" and "or": Use De Morgan's Law

- $\neg(p \wedge q) \equiv$
- $\neg(p \vee q) \equiv$

Example 10. What is the negation of "$n < 10$ or n is odd"?

Negation of "there exists" and "for all":

- $\neg$(There exists something such that p) $\equiv$
- $\neg$(For all something, p)$\equiv$

Example 11.

- What is the negation of "For all real numbers x, x has a real square root"?

Negation of implication: Use the "Or form": $p \to q \equiv (\neg p) \vee q$.

- $\neg(p \to q) \equiv$

$\star$ **Note:** "$p \to q$", "$(\neg q) \to (\neg p)$", and "$(\neg p) \vee q$" are logically equivalent statements.

We can verify this by truth table:

p	q	$p \to q$	
T	T		
T	F		
F	T		
F	F		

Example 12. What is the negation of "A differentiable function is continuous. "?

⋆ Suggested exercises from the textbook: Section 0.1 # 5,6,7 (some are assigned in Homework 1).

Summary - fill in the blanks. Answers will be posted on Canvas.

- Negation of statements

Original statement	Negation of statement	Example of original	Example of negation
$p \wedge q$		"x is prime and x is even"	
$p \vee q$		"$2^3 = 6$ or $\sqrt[3]{4} = 2$"	
$\exists x, p$		"There is a real number x such that $x^2 = -1$"	
$\forall x, p$		"For all integer n, -1 is a factor of n"	
$p \to q$		"If 2 is even, then 6 is odd. "	

- Logically equivalent forms of implication: $p \to q \equiv \boxed{} \equiv \boxed{}$

Sec. 1.1: Truth Tables

We have seen a few examples of truth tables last week. Here are more examples.

Example 13. Construct a truth table for $p \vee r \to q$.
(Challenge: Show that $p \vee r \to q$ and $(p \to q) \wedge (r \to q)$ are logically equivalent.)

p	q	r	
T			
T			
T			
T			
F			
F			
F			
F			

Example 14. Find the truth value of $p \vee r \to q$ if p, q are true, and r is false.

p	q	r	

Definition. A compound statement is a

- **tautology**, if it is always $\boxed{}$, regardless of the truth values assigned to its variables. Example:

p	$\neg p$	$p \vee \neg p$
T	F	
F	T	

- **contradiction**, if it is always $\boxed{}$, regardless of the truth values assigned to its variables. Example:

p	$\neg p$	$p \wedge \neg p$
T	F	
F	T	

Sec. 0.2: Proofs in Mathematics

Many mathematical theorems are statements that a certain implication is true.

In an implication $p \to q$, we say that

- p is the **hypothesis**, and q is the **conclusion**.
- p is a **sufficient** condition of q.
- q is a **necessary** condition of p.

If $p \leftrightarrow q$, we say that p is a **necessary and sufficient** condition of q.

Methods of Proof

Direct proof of an implication: Assume that the hypothesis is true, and use the hypothesis and other known true statements and definitions to deduce that the conclusion is true.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 15. Consider the statement: "$0 < x < 1 \to x^2 < 1$."

(a) What is the hypothesis?

(b) What is the conclusion?

(c) **Wrong proof:**

(d) **Correct proof:**

(1) Assume that ______ is true.

(2) Then ______ is true and ______ is true

(3) Multiplying both sides of ______ by ______, we get ______.

(4) Using (2), we can make the chain of inequalities ______, which implies $x^2 < 1$.

Disprove by Counterexample: To show that an implication or a "for all" statement is false, it is enough to find a single case (a counterexample) that the statement does not hold.

Example 16. Consider the statement again: "$0 < x < 1 \to x^2 < 1$." Disprove the converse of the statement.

Prove by cases: Sometimes the hypothesis can be breaking into a nubmer of cases. We may prove the implication by showing that the conclusion is true for each cases.

Example 17. Prove that for any integer n, $n^2 + n$ is even.

- **Case 1:** Let n be an ______ integer.

  Then n^2 is the product of two even integers, which is ______.

  Thus, $n^2 + n$ is the sum of two even integers, which is ______.

- **Case 2:** Let n be an $\boxed{}$ integer.

  Then n^2 is the product of two odd integers, which is $\boxed{}$.

  Thus, $n^2 + n$ is the sum of two odd integers, which is $\boxed{}$.

Proving double implication: To prove a double implication $p \leftrightarrow q$:

1. Prove $p \rightarrow q$.
2. Prove $p \leftarrow q$.

Example 18. Prove that for all real numbers a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$.

($\leftarrow$):

($\rightarrow$):

$\star$ Suggested exercises from the textbook:
Section 1.1 # 1, 2, 4, 6, 9, 11 (some are assigned in Homework 2).
Section 0.2 # 1, 2, 3, 4 (all are assigned in Homework 1).

Summary - fill in the blanks. For Section 1.1, review the examples in Lecture 01, 02, 03.

If $p \rightarrow q$, then the hypothesis is $\boxed{}$ and the conclusion is $\boxed{}$.
p is a $\boxed{}$ condition of q, and q is a $\boxed{}$ condition of p.

Method of Proof

- Direct proof: Assume that $\boxed{}$ is true, and show that $\boxed{}$ is true.
- Disprove by counterexample
- Prove by cases
- Proving double implication: Show $\boxed{}$ and $\boxed{}$.
- Prove the contrapositive (next time!)
- Prove by contradiction (next time!)

Sec. 0.2: Proofs in Mathematics (cont.)

Prove the contrapositive: Since an implication is logically equivalent to its contrapositive, we may also prove the contrapositive instead (if it's easier!).

Example 19. Show that if x^2 is even, then x is even.

(a) What is the contrapositive?

(b) Prove the contrapositive of the statement.

Prove by contradiction: To prove a statement $\mathcal{A}$ is true:

1. Assume that $\neg\mathcal{A}$ is true.
2. If this leads to a false statement (a contradiction), then $\neg\mathcal{A}$ is false. Thus, $\mathcal{A}$ must be true.

Example 20. Prove the statement: $\mathcal{A}$: "There is no largest integer. "

1. Assume $\neg\mathcal{A}$: "There is a largest integer. "
2. Denote this largest integer by N. Then $N \geq m$ for all integers m.
   Consider $\boxed{}$, which is also an integer because it is the sum of two integers.
   Then $\boxed{}$, which contradicts our assumption. Thus $\neg\mathcal{A}$ is false, so $\mathcal{A}$ is true.

Sec. 2.1: Sets

A **set** is a collection of objects. The objects are called "elements" or "members" of the set.

Example 21.

- $B = \{\text{ice cream, coffee, donut}\}$
- $\{3, 1, 2\}$, $\{1, 2, 3, 3\}$, $\{x, y, \{1\}\}$

Notation:

- "$\in$":
- "$\notin$":
- Set builder notation: $\{x | x \text{ has some properties}\}$, $\{(\text{expression}) | \text{ (expression) has some properties}\}$.

Exercise 22. List the distinct elements of the set $\{x + y \mid x \in \{-1, 0\}, y \in \{1, 2\}\}$.

Sets we will frequently use:

- The set of **natural numbers**:
- The set of **integers**:
- The set of **even integers**:
- The set of **odd integers**:
- The set of **rational numbers**:
- The set of **real numbers**:
- The set of **complex numbers**:

Definition. Two set A and B are **equal**, written as $\boxed{\qquad\qquad}$, if and only if they have the same elements.

Note: This includes the case that neither set contains any element.

Definition. The unique set with no elements is called the **empty set**, denoted by $\boxed{\qquad}$.

Example 23. $\{x \in \mathbb{R} | x^2 + 1 = 0\}$, $\{r \in \mathbb{Q} | r^2 = 2\}$

Definition. Let A and B be two sets.

- A is a **subset** of B, written $\boxed{\qquad\qquad}$, if and only if every element of A is also an element of B.
- If $A \subseteq B$ but $A \neq B$, then A is a **proper subset** of B, written $\boxed{\qquad\qquad}$.

Example 24. The following statements are true

1. $a \in \{a, b, \{c\}\}$
2. $b \in \{a, b, \{c\}\}$
3. $c \notin \{a, b, \{c\}\}$
4. $\{c\} \in \{a, b, \{c\}\}$
5. $\{a, \{c\}\} \subseteq \{a, b, \{c\}\}$
6. $\{a, \{c\}\} \subsetneqq \{a, b, \{c\}\}$
7. $\{a, b, c\} \nsubseteq \{a, b, \{c\}\}$

Example 25. $\mathbb{N} \subsetneqq \mathbb{Z} \subsetneqq \mathbb{Q} \subsetneqq \mathbb{R} \subsetneqq \mathbb{C}$. (Exercise: Show that each "$\neq$" is true (find examples))

Proposition 26. *For any set A, $A \subseteq A$ and $\emptyset \subseteq A$.*

Proof.

- $A \subseteq A$:
- $\emptyset \subseteq A$ (prove by contradiction):

$\square$

Proposition 27. *If A and B are sets, then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.*

⋆ In general, to show that two sets are equal, we can show that $A \subseteq B$ and $B \subseteq A$. We will see examples in later sections.

Exercise 28. Determine the truth value of the following statements.

1. $\{a\} \subseteq \{x, y, a\}$
2. $\{a\} \in \{x, y, a\}$
3. $\{a\} \subseteq \{x, y, \{a\}\}$
4. $\emptyset \in \{\emptyset\}$
5. $\emptyset \subseteq \{\emptyset\}$
6. $\{\emptyset\} \subseteq \{\emptyset\}$
7. $\{\emptyset\} \subseteq \{\{\emptyset\}\}$
8. $\{\emptyset\} \in \{\emptyset\}$
9. $\emptyset \subseteq \{\{\emptyset\}\}$
10. $\emptyset \in \{\{\emptyset\}\}$
11. $\emptyset \subseteq \{x, y, \emptyset\}$
12. $\emptyset \in \{x, y, \emptyset\}$

⋆ Suggested exercises from the textbook:
Section 2.1 # 1, 3, 4, 5, 7, 12 (some are assigned in Homework 2)

Summary - fill in the blanks. Answers will be posted on Canvas.

- A **set** is a collection of objects. The objects are called ______ or ______ of the set.
- If $A = \{1, 2, 3\}$, then 1 ______ A, and 4 ______ A.
- The symbol $\emptyset$ denotes ______.
- A is a **subset** of B, written ______, if and only if ______.
- A is a **proper subset** of B, written ______, if and only if ______.
- For any set A, ______ $\subseteq A$ and ______ $\subseteq A$.
- Sets A and B are equal if and only if A ______ B and B ______ A.
- Fill in the blanks with $\mathbb{C}, \mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}$: ______ $\subsetneq$ ______ $\subsetneq$ ______ $\subsetneq$ ______ $\subsetneq$ ______.

Sec. 2.1: Sets (cont.)

Definition. The **power set** of a set A, denoted by $P(A)$, is the set of all subset of A.

Using set builder notation: $\boxed{}$

Example 29.

- If $A = \emptyset$, then $P(A) =$
- If $A = \{a\}$, then $P(A) =$
- If $A = \{a, b\}$, then $P(A) =$
- If $A = \{a, b, c\}$, then $P(A) =$
- If A has n elements, then $P(A)$ has $\boxed{}$ elements. (will prove in Sec. 5.1)

Sec. 2.2: Operation on Sets

Let A and B be two sets.

Definition: Union and intersection

- The **union** of A and B, written $\boxed{}$, is the set of elements in A $\boxed{}$ B.
  Venn diagram:
- The **intersection** of A and B, written $\boxed{}$, is the set of elements in A $\boxed{}$ B.
  Venn diagram:

Example 30. Let $A = \{a, b, c\}$ and $B = \{a, x, y, b\}$. Then

1. $A \cup B =$
2. $A \cap B =$
3. $A \cup \{\emptyset\} =$
4. $B \cap \{\emptyset\} =$

Example 31. For any set A, $A \cup \emptyset = \boxed{}$, $A \cap \emptyset = \boxed{}$.

Properties of union and intersection

Associativity

- $(A_1 \cup A_2) \cup A_3 = \boxed{}$
- $(A_1 \cap A_2) \cap A_3 = \boxed{}$
- $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n = \boxed{}$
- $A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n = \boxed{}$

Distributivity

- $A \cap (B \cup C) =$
- $A \cup (B \cap C) =$

Example 32. Find a counterexample to disprove that $A \cap (B \cup C) = (A \cap B) \cup C$.
(Exercise: Disprove $A \cup (B \cap C) = (A \cup B) \cap C$.)

Example 33. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(Exercise: Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.)

Definition: Set difference and complement

- The **set difference** of A and B, written $\boxed{}$, is the set of elements of A that is not in B.
  Venn diagram:
- The **complement** of A, written $\boxed{}$, is the set $\boxed{}$ where U is an universal set made clear by the context.
  Venn diagram:
- **Note:**

Example 34.

1. $\{a, b, c\} \setminus \{a, b\} =$
2. $\{a, b, c\} \setminus \{a, x\} =$
3. $\{a, b, \emptyset\} \setminus \emptyset =$
4. $\{a, b, \emptyset\} \setminus \{\emptyset\} =$

Example 35. Let $U = \mathbb{R}$.

1. If $A = (-\infty, 1) \cup (1, 3]$, then $A^c =$
2. If $A = \{x \in \mathbb{R} | x^2 \leq -1\}$, then $A^c =$

De Morgan's Law

- $(A \cup B)^c = \boxed{}$
- $(A \cap B)^c = \boxed{}$

Definition: Symmetric difference

- The **symmetric difference** of A and B, written $\boxed{}$, is the set of elements that are in A or B, but not in both.
  Venn diagram:
- Other ways to express $A \oplus B$:

Example 36.

1. $\{a, b, c\} \oplus \{x, y, a\} =$
2. $\{a, b, c\} \oplus \emptyset =$
3. $\{a, b, c\} \oplus \{\emptyset\} =$

Definition: Cartesian product

- The **Cartesian product** of A and B is the set $\boxed{}$.
- $(a_1, b_1) = (a_2, b_2) \leftrightarrow$
- In general, $A_1 \times A_2 \times \cdots A_n =$

Example 37. Let $A = \mathbb{R}$. Then $A \times A =$

Example 38. Let $A = \{a, b\}$, $B = \{x, y\}$. Then

1. $A \times B =$
2. $B \times A =$

Example 39. Let A, B, and C be sets. Prove that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.
(Exercise: Show "$\supseteq$")

$\star$ Suggested exercises from the textbook: (some are assigned in Homework # 2)
Section 2.1 for power set: # 14
Section 2.2: # 1, 2, 4, 6, 12, 15, 19, 27, 28, 29

Summary - fill in the blanks. Answers will be posted on Canvas.

- The **power set** of A is $\boxed{}$.
- Let A, B be two sets and U be the universal set. Complete the table about operations on sets:
  *For "Example", use $A = \{1, 2\}, B = \{2, 3\}, U = \{1, 2, 3, 4\}$

Symbol	Name	Definition	Venn diagram	Example*
$A \cup B$		$x \in A \cup B \leftrightarrow$		
$A \cap B$		$x \in A \cap B \leftrightarrow$		
$A \setminus B$		$x \in A \setminus B \leftrightarrow$		
A^c				
$A \oplus B$		$x \in A \oplus B \leftrightarrow$		
$A \times B$		$A \times B = \{ \qquad \}$	n/a	

- $(A_1 \cup A_2) \cup A_3 = \boxed{}$, $(A_1 \cap A_2) \cap A_3 = \boxed{}$
- $A \cap (B \cup C) = \boxed{}$, $A \cup (B \cap C) = \boxed{}$
- $(A \cup B)^c = \boxed{}$, $(A \cap B)^c = \boxed{}$

Sec. 2.3: Binary Relations

Let A and B be two sets.

Definition.

- A **binary relation from** A **to** B is a subset of $A \times B$.
- A **binary relation on** A is a subset of $A \times A$.

Notation:

Example 40.

- $A =$All students in UD, $B =$All courses in UD, $\mathcal{R} = \{(a, b) \in A \times B|$ student a is taking course $b\}$.
- $A = \mathbb{Z}$, $B = \mathbb{N}$, $(a, b) \in \mathcal{R} \leftrightarrow |a| = b$.
- $A = \mathbb{N}$, $\mathcal{R} = \{(a, b) \in \mathbb{N} \times \mathbb{N}| \frac{a}{b}$ is an integer$\}$.

Properties of binary relations on a set A (so they are subsets of $A \times A$).

Definition: Reflexive
A binary relation $\mathcal{R}$ on a set A is **reflexive** if and only if

Example 41. Prove or disprove whether each of the binary relations $\mathcal{R}$ is reflexive.

- $A = \mathbb{N}, \mathcal{R} = \{(a, b) \in \mathbb{N}^2 | \frac{a}{b} \in \mathbb{N}\}$.
- $A = \mathbb{R}, \mathcal{R} = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 > 0\}$.

Definition: Symmetric
A binary relation $\mathcal{R}$ on a set A is **symmetric** if and only if

Example 42. Prove or disprove whether each of the binary relations $\mathcal{R}$ is symmetric.

- $A = \mathbb{R}, \mathcal{R} = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$.

- $A = \mathbb{R}, \mathcal{R} = \{(x, y) \in \mathbb{R}^2 | x^2 \geq y\}$.

Example 43. Let $A = \{1, 2, 3\}$. Give an example of a relation $\mathcal{R}$ on A that is reflexive and symmetric.

> **Definition: Antisymmetric**
> A binary relation $\mathcal{R}$ on a set A is **antisymmetric** if and only if

Example 44. Prove or disprove whether each of the binary relations $\mathcal{R}$ is antisymmetric.

- Let S be a set, $A = P(S)$, $\mathcal{R} = \{(X, Y) \in A^2 | X \subseteq Y\}$.

- $A = \mathbb{R}^2$, $\mathcal{R} = \{((x, y), (u, v)) \in \mathbb{R}^2 \times \mathbb{R}^2 | x^2 + y^2 = u^2 + v^2\}$.

> **Definition: Transitive**
> A binary relation $\mathcal{R}$ on a set A is **transitive** if and only if

Example 45. Prove or disprove whether each of the binary relations $\mathcal{R}$ is transitive.

- $A = \mathbb{Z}, \mathcal{Z} = \{(a, b) \in \mathbb{Z}^2 | \frac{a}{b} \in \mathbb{Z}\}$.

- $A = \mathbb{R}, \mathcal{R} = \{(x, y) \in \mathbb{R}^2 | x^2 \geq y\}$.

Example 46. Let $A = \{1, 2, 3\}$. Give an example of a relation $\mathcal{R}$ on A that is transitive.

Exercise 47. Prove or disprove whether each of the binary relations $\mathcal{R}$ is reflexive, symmetric, antisymmetric, or transitive.

- $A = \{1, 2, 3, 4\}$, $\mathcal{R} = \{(1, 1), (1, 2), (2, 1), (3, 4)\}$.

- $A = \mathbb{Z}$, $(a, b) \in \mathcal{R} \leftrightarrow ab \geq 0$.

⋆ Suggested exercises from the textbook:
Section 2.3 # 3, 5, 7, 9, 10 (some are assigned in Homework # 2)

Summary - fill in the blanks.

Let A and B be two sets.

- A **binary relation from A to B** is $\boxed{}$
- A **binary relation on A** is $\boxed{}$

Let $\mathcal{R}$ be a binary relation on a set A. Then

- $\mathcal{R}$ is reflexive $\leftrightarrow$
- $\mathcal{R}$ is symmetric $\leftrightarrow$
- $\mathcal{R}$ is antisymmetric $\leftrightarrow$
- $\mathcal{R}$ is transitive $\leftrightarrow$

Sec. 2.4: Equivalence Relations

Definition. An **equivalence relation** on a set A is a binary relation $\mathcal{R}$ (or written as $\sim$) on A that is ____, ____, and ____.
Note: If $a \sim b$ and $\sim$ is an equivalence relation, then we say

Example 48. Show that each of the following relation is an equivalence relation on A.

1. $A = \mathbb{R}$, $(a, b) \in \mathcal{R}$ if and only if $a = b$.

2. $A =$All people in the world, $a \sim b \leftrightarrow a, b$ have the same birthday. (exercise)

Definition. If $a, b \in \mathbb{Z}$ and $b \neq 0$. Then a **is divisible by** b, written as ____, if and only if ____ for some $n \in \mathbb{Z}$. (Equivalently, "b divides a", "a is a multiple of b".)

Example 49. Show that $\sim$ is an equivalent relation on A if

$$A = \mathbb{Z},\ a \sim b \text{ if and only if } a - b \text{ is divisible by } 3.$$

Remark: This relation is called ____.
And if $a \sim b$ in this case, we say ____.

Definition. Let $\sim$ be an equivalence relation on a set A.

- The **equivalence class** of an element $a \in A$ is the set ____
- The set of all equivalence classes, written ____, is called the **quotient set of** A **mod** $\sim$

Example 50.

1. If $\sim$ is the equivalence in relation in Example 1 # 2, then

$$A/\sim = \big\{\{\text{people born in Jan 1}\}, \{\text{people born in Jan 2}\}, \ldots, \{\text{people born in Dec 31}\}\big\}.$$

2. Let $\sim = \{(a, b) \in \mathbb{Z}^2 | a - b \text{ is divisible by } 3\}$. Determine $\mathbb{Z}/\sim$.

   $\star$ **Notation:** for $n, r \in \mathbb{N}$, $n\mathbb{Z} + r =$

3. In general, for $n \in \mathbb{N}$, **congruence mod** n: $A = \mathbb{Z}$, $a \sim b$ if and only if $a - b$ is divisible by n.
   $\mathbb{Z}/\sim = \boxed{\qquad\qquad\qquad}$

Observations about equivalence classes

1. If $\sim$ is an equivalence relation on A, then by reflexivity, $\boxed{\qquad\qquad\qquad}$
2. From the congruence mod 3 example, we see that $\overline{0} = \overline{3}$, $\overline{1} = \overline{4}$,...
   In general, for any equivalence relation $\sim$ on A, if $\boxed{\qquad\qquad}$ then $\boxed{\qquad\qquad}$.
   Is the converse true?
3. Equivalence classes do not overlap!

Proposition 51. *Let $\sim$ be an equivalence relation on a set A, and $a \in A$. Then, for any $x \in A$,*

$$x \sim a \leftrightarrow \overline{x} = \overline{a}.$$

Proposition 52. *Let $\sim$ be an equivalence relation on a set A, and $a, b \in A$. Then $\overline{a}$ and $\overline{b}$ are either the same or disjoint.*

Proof.

Definition. A **partition** of a set A is a collection of disjoint nonempty subsets of A whose union is A. These disjoint sets are called **cells** or **blocks**. The cells are said to **partition** A.

Notation:

Theorem. The equivalence classes associated with an equivalence relation on a set A form a partition of A.

Example 53.

1. Let $\sim = \{(a,b) \in \mathbb{Z}^2 | a-b \text{ is divisible by } 2\}$. Then $\mathbb{Z}/\sim = \{\overline{0}, \overline{1}\} = \{2\mathbb{Z}, 2\mathbb{Z}+1\}$.
2. If A = a set of standard 52-card deck. Then A may be partitioned by the suits (♠♡♢♣):
   - Define the equivalence relation $\sim$ by ______
   - For any $a \in A$, $|\overline{a}| =$ ______
   - $|A/\sim| =$ ______

Example 54. In $\mathbb{R}^2$, define $\sim$ by $(x,y) \sim (u,v)$ if $x^2 + y^2 = u^2 + v^2$. What does the quotient set looks like geometrically? (Exercise: Show that $\sim$ defines an equivalence relation on $\mathbb{R}^2$)

$\star$ Suggested exercises from the textbook:
Section 2.4 # 4–12, 16, 18 (some are assigned in Homework # 3)

Summary - fill in the blanks.

- An **equivalence relation** on a set A is a binary relation $\mathcal{R}$ on A that is ______, ______, and ______.
- The **equivalence class** of an element $a \in A$ is ______.
- The **quotient set of A mod $\sim$**, written ______, is ______.
- **Improtant proposition and theorem:** Let $\sim$ be an equivalence relation on A. Then
  (a) For any $x \in A$, $\overline{x} = \overline{a} \leftrightarrow$ ______
  (b) If $a, b \in A$, then $\overline{a}$ and $\overline{b}$ are either ______ or ______.
  (c) The equivalence classes of $\sim$ form a ______ of A.

Sec. 3.1: Basic Terminology (Functions)

Let A, B be two sets.

> **Definition.** A **function** from A to B is a binary relation f from A to B such that

Example 55. Let $A = \{1, 2, 3\}$, $B = \{x, y\}$. Which of the following defines a function from A to B?

1. $f_1 = \{(1, x), (2, x), (3, y)\}$
2. $f_2 = \{(1, y), (3, x)\}$
3. $f_3 = \{(1, x), (2, y), (2, x), (3, y)\}$

Notation: If $f = \{(1, x), (2, x), (3, y)\}$ is a funciton from A to B, then we can also write

Example 56. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

> **Definition.** Let $f : A \to B$. Then
>
> - The **domain of** f, written $\boxed{\qquad\quad}$, is $\boxed{\qquad}$.
> - The **target (codomain) of** f, is $\boxed{\qquad}$.
> - The **range (image) of** f, written $\boxed{\qquad\quad}$, is

Example 57. Let $A = \{1, 2, 3, 4\}$, $B = \mathbb{Z}$, $f = \{(x, x^2) | x \in A\}$.

1. $\mathrm{dom} f =$
2. target of $f =$
3. $\mathrm{rng} f =$

> **Definition.** Let $f : A \to B$ be a function. Then
>
> - f is **one-to-one (injective)** if and only if
>   In symbols:
> - f is **onto (surjective)** if and only if
>   In symbols:
> - f is **bijective** if and only if

Example 58.

domain	$A_1 = \{1, 2, 3, 4\}$	$A_2 = \{1, 2, 3\}$	$A_3 = \{1, 2, 3\}$
target	$B_1 = \{x, y, z\}$	$B_2 = \{w, x, y, z\}$	$B_3 = \{x, y, z\}$
function	$f_1 = \{(1, x), (2, y), (3, z), (4, y)\}$	$f_2 = \{(1, w), (2, y), (3, x)\}$	$f_3 = \{(1, z), (2, y), (3, x)\}$
picture			
one-to-one?			
onto? (range=?)			
bijective?			

Example 59. Let $A = \mathbb{R}$, and $f : A \to A$ such that $f(x) = 2x - 3$. Is f one-to-one? Is f onto? Is f bijective?

Exercise 60. Let $A = \mathbb{Z}$, and $f : A \to A$ such that $f(x) = 2x - 3$. Is f one-to-one? Is f onto? Is f bijective?
(Exercise: What if $A = \mathbb{N}$ instead?)

Definition. Let A be a set. The **identity function on** A is the function

(Exercise: Is the identity function one-to-one? onto? bijective?)

Definition. Let x be a real number.

- The **floor of** x, written $\boxed{}$, is
- The **ceiling of** x, written $\boxed{}$, is

Example 61.

- $\lfloor 1.05 \rfloor =$, $\lfloor 2.99 \rfloor =$, $\lfloor 3 \rfloor =$, $\lfloor -1.05 \rfloor =$
- $\lceil 1.05 \rceil =$, $\lceil 2.99 \rceil =$, $\lceil 3 \rceil =$, $\lceil -1.05 \rceil =$

Example 62. Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $f(n, m) = \lfloor \frac{n}{m} \rfloor + 1$. Is f one-to-one? (Exercise: Is f onto?)

⋆ Suggested exercises from the textbook:
Section 3.1 # 4, 12-17, 22, 29, 31, 34 (some are assigned in Homework # 3)

Summary - fill in the blanks.

- A **function** from A to B is a $\boxed{}$
  such that $\boxed{}$
- If $f : A \to B$, then
  - **dom**f=$\boxed{}$, **target of** f=$\boxed{}$, **rng**f=$\boxed{}$
  - f is **one-to-one** $\leftrightarrow$
  - f is **onto** $\leftrightarrow$
  - f is **bijective** $\leftrightarrow$
- The **identity function on a set** A is
- If $x \in \mathbb{R}$, then $\lfloor x \rfloor =$
  $\lceil x \rceil =$

Sec. 3.2: Inverses and Composition

Definition. Let $f : A \to B$ be a funciton.

- f **has an inverse** if and only if the relation obtained by reversing the ordered pairs of f is a function $B \to A$.
- If f has an inverse, then the **inverse of** f is

Note:

Example 63. Let $A = \{1, 2, 3, 4\}$, $B = \{w, x, y, z\}$, and $f = \{(1, w), (2, x), (3, y), (4, z)\}$.
Then $f^{-1} =$

Proposition 64. *A function f has an inverse if and only if f is one-to-one and onto.*

Example 65. Let $A = \mathbb{R}$ and $f(x) = 2x - 3$. We showed that f is bijective in Sec. 3.1, so f has an inverse f^{-1}. Find f^{-1}.

Example 66. Let $f : \mathbb{R}^+ \to \mathbb{R}$ and $f(x) = \log_3 x$. Find f^{-1}, and the domain and range of f^{-1}.

Definition. Let $f : A \to B$ and $g : B \to C$. Then the **composition of** g **and** f is the function

Note: For $g \circ f$ to exist, we must have $\boxed{}$.

Example 67. Let $A = \{1, 2, 3\}$, $B = \{x, y\}$, $C = \{a, b\}$. Define $f : A \to B$ by $f = \{(1, x), (2, x), (3, y)\}$, $g : B \to C$ by $g = \{(x, b), (y, a)\}$. Find $g \circ f$.

Example 68. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x - 3$, and $g : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ by $g(x) = \frac{x}{x-1}$. Then

$(g \circ f)(x) =$

$(f \circ g)(x) =$

Definition. Two functions f, g are **equal** if and only if they have the same domain, same target, and $f(a) = g(a)$ for all a in the domain.

Example 69. If $f : A \to A$, show that $f \circ \mathrm{id}_A = f$. (Exercise: Show that $\mathrm{id}_A \circ f = f$.)

Proposition 70. *Composition of functions is an* ***associative*** *operation.*

Proposition 71. *If $f : A \to B$, $g : B \to A$, then f, g are inverses of one another if and only if*

Exercise 72. Define $f, g, h : \mathbb{R} \to \mathbb{R}$ by $f(x) = x + 2$, $g(x) = \frac{1}{x^2+1}$, $h(x) = 3$. Then

$(h \circ g \circ f)(x) =$

$(g \circ h \circ f)(x) =$

$(g \circ f \circ f^{-1})(x) =$

$\star$ Suggested exercises from the textbook:
Section 3.2 # 3, 5, 6, 10, 11, 13 (some are assigned in Homework # 4)

Summary - fill in the blanks.

- A function f has an inverse $\leftrightarrow$ ______ (Proposition 2).
- If $f : A \to B$ has an inverse, then the inverse of f is ______ (relation notation).
- If f : ______ and g : ______, then the **composition of** g **and** f is the function ______ : ______ defined by ______ = ______ for all $a \in A$.
- If $f : A \to A$, then $f \circ \mathrm{id}_A =$ ______ = ______
- If $f : A \to B$ has an inverse, then $f \circ f^{-1} =$ ______, and $f^{-1} \circ f =$ ______
- Associativity: $(f \circ g) \circ h =$ ______

Sec. 3.3: One-to-One Correspondence and the Cardinality of a Set
⋆ We skip §3.3 for the assesment, but the following definitions may be used for the later sections.

Definition. A function $f : A \to B$ is called an **one-to-one correspondence** if and only if it is one-to-one and onto (bijective).

Example 73. Let $A = \{1, 2, 3\}$, $B = \{x, y, z\}$. Define $f : A \to B$ by $f(1) = x, f(2) = y, f(3) = z$. Then we say

- f is an one-to-one correspondence between A and B (from A to B).
- A and B are in one-to-one correspondence.

Definition. Let A be a set.

- A is **finite** if either
  (a) $A = \emptyset$.
  (b) A is in one-to-one correspondence with the set $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.
- The **cardinality** of A, written $|A|$, is
  (a) $|A| = 0$, if $A = \emptyset$.
  (b) $|A| = n$, if A is in one-to-one correspondence with the set $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.
- A is **infinite** if it is not finite.

Chapter 6: Principles of Counting

Sec. 6.1: The Principle of Inclusion-Exclusion

The **Principle of Inclusion-Exclusion** is a counting technique that computes the number of elements satisfying multiple properties, while guaranteeing elements satisfying more than one property are not counted twice.

Example 74. There are 350 job applicants, with the following majors:

Business: 140

Computer Science: 220

Both: 39

(a) How many applicants major in either Business or Computer Science ?

In general, $|A \cup B| =$

(b) What if we didn't know the number of those majoring in both? How many could there be? (How big could $|B \cap C|$ be?)

In general, $|A \cap B| \leq$

(c) How many applicants are Business majors, but not CS majors?

In general, $|A \setminus B| =$

(d) How many applicants are majoring in neither CS nor Business?

In general, $|A^c| =$

(e) How many applicants are either CS majors who are not Business majors, or Business majors who are not CS majors?

In general, $|A \oplus B| =$

Proposition 75. *Let A and B be subsets of a finite universal set U. Then*

(a) $|A \cup B| = |A| + |B| - |A \cap B|$

(b) $|A \cap B| \leq \min\{|A|, |B|\}$

(c) $|A \setminus B| = |A| - |A \cap B| \geq |A| - |B|$

(d) $|A^c| = |U| - |A|$

(e) $|A \oplus B| = |A \setminus B| + |B \setminus A| = |A| + |B| - 2|A \cap B| = |A \cup B| - |A \cap B|$

(f) $|A \times B| = |A||B|$

Proposition 76. *(18[BB]) If $a, b \in \mathbb{N}$, then the number of natural numbers less than or equal to a and divisible by b is $\lfloor \frac{a}{b} \rfloor$*

Example 77. (a) How many integers between 1 and 100 are divisible by at least one of 2 or 3?

(b) How many integers between 1 and 100 are divisible by 2 and 3, but not by 5?

Example 78. Suppose A, B, and C are finite sets. Find a formula of $|A \cup B \cup C|$.

Theorem 79 (The Principle of Inclusion-Exclusion)**.** *Let $A_1, A_2, \ldots, A_n$ be a finite number of finite sets. Then*

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_n| = &+ \sum_{i=1}^{n} |A_i| \\ &- \sum_{i<j} |A_i \cap A_j| \\ &+ \sum_{i<j<k} |A_i \cap A_j \cap A_k| \\ &- \sum_{i<j<k<\ell} |A_i \cap A_j \cap A_k \cap A_\ell| \\ &\cdots (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.\end{aligned}$$

Exercise 80. You go to the UD Bookstore looking for a mug to buy as a gift. You find 33 total kinds of mugs, of which:

- 13 have the Blue Hen logo
- 11 contain both blue and yellow
- 23 say "Delaware"

You also see that 13 have at least two of these properties, while 4 have all three.

(a) How many mugs have at least one property?
$|B \cup C \cup D| = |B| + |C| + |D| - |B \cap C| - |C \cap D| - |B \cap D| + |B \cap C \cap D|$

⋆ "Have at least two properties" means:

(b) How many have none?

(c) How many have exactly one property?

⋆ Suggested exercises from the textbook:
Section 6.1 (two or three sets:) # 3, 4, 7, 11;
(four sets:) # 9, 10, 15 (some are assigned in Homework # 4)

Summary - fill in the blanks.

⋆ Venn diagram from Sec 2.2 might be useful for the following formulas.

- If A and B be subsets of a finite universal set U, then
  (a) $|A \cup B| =$
  (b) $|A \cap B| \leq$
  (c) $|A \setminus B| =$
  (d) $|A^c| =$
  (e) $|A \oplus B| =$
  (f) $|A \times B| =$
- The Principle of Inclusion-Exclusion - 3 sets version:
  $|A \cup B \cup C| =$

Sec. 6.2: The Addition and Multiplication Rules

Definition.

- Two sets A and B are **disjoint** if and only if $A \cap B = \emptyset$.
  $\star$ So in this case, $|A \cap B| = \boxed{}$, and $|A \cup B| = |A| + |B| - |A \cap B| = \boxed{}$
- Sets $A_1, A_2, \ldots, A_n$ are **pairwise disjoint** if and only if $\forall i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, $A_i \cap A_j = \emptyset$.

Proposition 81 (The Principle of Inclusion-Exclusion for disjoint sets)**.** *If $A_1, A_2, \ldots, A_n$ are pairwise disjoint finite sets, then*

$$|A_1 \cup A_2 \cup \cdots \cup A_n| =$$

Example 82.

- The sets $A_1 = \{1, 2\}, A_2 = \{3, 4\}, A_3 = \{5, 6, 7\}$ are pairwise disjoint.
  So $|A_1 \cup A_2 \cup A_3| =$
- The sets $B_1 = \{1, 2\}, B_2 = \{3, 4\}, B_3 = \{4, 5, 6\}$ are NOT pairwise disjoint.
  So $|B_1 \cup B_2 \cup B_3| \neq |B_1| + |B_2| + |B_3|$.

Proposition 83. *If $A_1, A_2, \ldots, A_n$ are finite sets, then*

$$|A_1 \times A_2 \times \cdots \times A_n| =$$

Definition.

- An **event** is a possible outcome or result of any experiment or process.
  $\star$ We want to think of an event be corresponding to a set A, where the elements in A represents the ways the event can occur.
- Events are **mutually exclusive** if no two of them can occur together.
  $\star$ The corresponding sets $A_1, A_2, \ldots, A_n$ are

Example 84. You are making a plan for a Friday night, say the three-hour window from 7 to 10 pm. You are invited to two parties, but also have three movies on the list that you would like to watch. Meanwhile, the four courses that you are taking all have homework due next week that you should work on.
How many ways can you spend your Friday night if...

(a) you would like to commit to only one activity for the three hours?

(b) you would like to do the homework of one course from 7-8, then go to one party from 8-9 to say hi, and then go back to watch one movie for the rest of the night?

> **Theorem 85** (Addition Rule)**.** *The number of ways in which precisely one of a collection of mutually exclusive events can occur is the sum of the numbers of ways in which each event can occur.*

> **Theorem 86** (Multiplication Rule)**.** *The number of ways in which a sequence of events can occur is the product of the numbers of ways in which each individual event can occur.*

Example 87. In how many ways can you get a total of six when rolling two dice?

Example 88. If a set A has n elements, then A has 2^n subsets, i.e., $|P(A)| = 2^n$. Explain why.

Example 89. (a) How many numbers in the range 1000-9999 do not have repeated digits?

(b) How many even numbers in the range 1000-9999 do not have repeated digits?

Exercise 90. Consider Example 4 again. How many ways can you spend your Friday night if you would like to do the homework of one course from 7-8, and relax from 8-10 by either going to one party or watching one movie?

$\star$ Suggested exercises from the textbook:
Section 6.2 # 4, 12, 15, 16, 17, 19 (some are assigned in Homework # 4)

Summary - fill in the blanks.

$\star$ The best way to study this section is by doing examples and exercises.
Let $A_1, A_2, \ldots, A_n$ be finite sets.

- $A_1, A_2, \ldots, A_n$ are pairwise disjoint $\leftrightarrow$
- If $A_1, A_2, \ldots, A_n$ are pairwise disjoint, then $|A_1 \cup A_2 \cup \cdots \cup A_n| =$
- $|A_1 \times A_2 \times \cdots \times A_n| =$
- An **event** is
- Events are **mutually exclusive** if
- Addition rule:
- Multiplication rule:

Sec. 6.3 The Pigeonhole Principle

Theorem 91 (The Pigeonhole Principle). *If n objects (pigeon) are put into m boxes (birdhouse) and $n > m$, then at least one box contains two or more of the objects.*

Theorem 92 (The Pigeonhole Principle (strong form)). *If n objects (pigeon) are put into m boxes (birdhouse) and $n > m$, then some box must contain at least* $\boxed{}$ *objects.*

Example 93. There are 27 people in the class.

(a) Can we all have different birth months?

(b) At least how many people in the class have birthdays in the same month?

Example 94. What is the minimum amount of people in a stadium to guarantee that at least 10 people will share the same exact birthday? (Include Feb 29)

Example 95. Given five points inside a square whose sides have length 2, prove that two are within $\sqrt{2}$ of each other.

(**Application...?** You are organizing an event on the north side of the Green in UD (between E Main St and E Delaware Ave). The dimension of this location is about 375 ft $\times$ 143 ft. At most how many people can you invite (mathematically, not legally), if you would like to make sure that everyone is at least 6 ft apart from each other?)

Example 96. You have two weeks to prepare for an exam. Suppose you plan to study for no more than 20 hours, and at least one hour a day and a whole number of hours each day. Show that there must be a period of consecutive days during which you will study exactly 12 hours.

Exercise 97. Show that in any list of 5 natural numbers, there must always exist a string of consecutive numbers that add up to a multiple of 5. (See Problem 13 in P.200)

Example 98. Show that in any group of six people, at least three must all know each other, or at least three must be mutual strangers.

Remark. The minimum number of people to guarantee that at least a know each other, or at least b are mutual strangers, is called the **Ramsey number** $R(a, b)$. We just showed that $R(3, 3) = 6$. In general, $R(a, b)$ is extremely difficult to determine–even the exact value of $R(5, 5)$ is still unknown.

$\star$ Suggested exercises from the textbook:
Section 6.3 # 2, 3, 5, 9, 12, 13, 16 (some are assigned in Homework # 5)

Chapter 7: Permutations and Combinations

Sec. 7.1: Permutation

Example 99. In how many ways can 7 people form a line?

Example 100. 7 people are attending a competition where the first, second, and third prize will be awarded. Assuming no ties, how many different outcomes are possible?

Definition.

(a) A **permutation** of a set of distinct objects is an arrangement of the objects in a line in some order.

(b) An r-**permutation** of n distinct objects is a permutation of r of the object $(r \leq n)$.

Proposition 101.

(a) The number of permutations of n distinct objects is $\boxed{}$

(b) The number of r-permutations of n distinct objects is $P(n, r)$, where $P(n, r)$ is defined by

- $P(n, 0) =$
- *For $r > 0$,* $P(n, r) =$

Example 102. Permutation of $\{a, b, c\}$: $abc, acb, bac, bca, cab, cba$

Exercise 103. Compute the following numbers.

$P(7, 3) =$ $P(25, 2) =$

$P(5, 5) =$ $P(19, 1) =$

Example 104. (Another interpretation of $P(n, r)$) In how many ways can you place 3 marbles of different colors (say red, green, and blue) into 7 numbered boxes, at most one marble per box?

Example 105. In how many ways can 7 people form a circle?

Example 106. 5 adults, 3 children, a dog and a cat are walking down a road one after another.

(a) In how many ways can this happen?

(b) In how many ways can this happen if the cat and the dog (in either order) lead the way?

(c) In how many ways can this happen if no two children are immediately following one another?

(d) In how many ways can this happen if the cat (and only the cat) is between Dr. Orchid and Prof. Plum?

(e) In how many ways can this happen if all children are together, but the cat and the dog cannot be immediately following one another?

Exercise 107. In how many ways can 7 people form a circle, if 3 of them can't be next to each other?

$\star$ Suggested exercises from the textbook:
Section 7.1 # 1-4, 6-8, 11, 12, 17 (some will be assigned in Homework # 6)

Summary - fill in the blanks.

You should also try and read the example problems in the textbook P.205 - P. 208.

- Definition of the symbol $P(n, r) =$
- Can you name at least two interpretation of $P(n, r)$?

Some basic permutation examples: ($n > 5$ in the following)

(a) In how many ways can n people form a line?

(b) In how many ways can n people form a circle?

(c) In how many ways can n people form a line, if 3 of them must be together (in any order)?

(d) In how many ways can n people form a line, if 3 of them do not want to be next to each other?

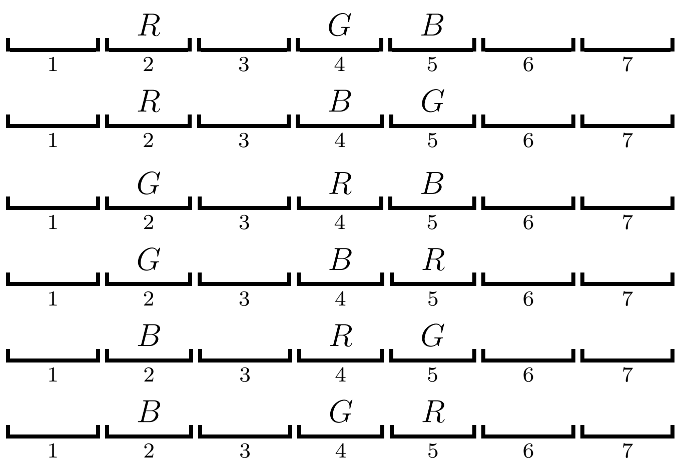
(e) In (c), (d), what if they form a circle instead?

(f) In how many ways can a club of n people select a panel of four officers consisting of a president, a vice-president, secretary, and treasurer?

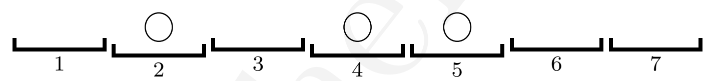
Sec. 7.2: Combinations

Example 108. Consider again the example of putting 3 marbles of red, green, and blue into 7 numbered boxes, at most one marble per box, from Sec. 7.1. What if the marbles are identical (same color)?

Distinguishable marbles ($P(7,3)$ ways)



Indistinguishable marbles



So we have

$$P(7,3) = (\text{number of ways to put 3 identical marbles in 7 boxes}) \times 3!$$

Therefore,

$$\begin{aligned}&\text{Number of ways to choose 3 boxes from 7}\\ &= \text{Number of ways to put 3 identical marbles in 7 boxes}\\ &= \frac{P(7,3)}{3!} =\end{aligned}$$

Definition.

- A **combination** of a set of objects is a subset of them.
- An r-**combination** of a set of n objects is a subset of r elements ($r \leq n$).

Proposition 109. *The number of r-combinations of n objects is $\binom{n}{r}$, where $\binom{n}{r}$ is defined by*

$$\binom{n}{r} =$$

Note:

- This symbol is called the **binomial coefficient**, and is read **"n choose r"** because it is the nubmer of ways to choose r objects from n.
- $$\binom{n}{r} = \binom{n}{n-r}$$

Example 110.

(a) The number of ways to choose 2 donuts from a dozen of different donuts is

(b) The number of ways to choose 3 people from a group of 25 is

Permutation vs. Combination

- **Permutations** are *orderings*, or re-arrangements of objects.
- **Combinations** are *selections*. Order dosen't matter.

Example 111. Ten friends are throwing a party. Six need to go out shopping while four stay and clean the house.

(a) In how many ways can the group decide who goes and who stays?

(b) What if Alice must be the driver?

(c) What if at least one of Bob or Chris must go?

(d) What if only one of Bob or Chris must go?

(e) What if Dave and Emily must stay?

Example 112. In how many ways can a 5-member committee be formed from 12 men and 20 women

(a) if exactly 3 women must be on the committee?

(b) if at least 3 women must be on the committee?

⋆ Suggested exercises from the textbook:
Section 7.2 # 1, 2, 5, 8-16 (some will be assigned in Homework # 6)

Summary - fill in the blanks.

See also Table 7.2 in the textbook P.214

- In how many ways can you choose r objects from n?
- If $0 \le r \le n$, then $\binom{n}{r} = \boxed{} = \frac{P(n,r)}{\boxed{}}$
- $\binom{n}{n-r} =$