MATH241-073D/101D Discussion Worksheet
A Quick Review for 2.8, 3.1–3.6, 4.1–4.3

Sec 2.8

- Definition of the derivative of $f(x)$: (a picture might be helpful)

$$f'(x) = \frac{\qquad\qquad}{\qquad\qquad}$$

- A function f is differentiable at a if the limit $f'(a)$ exists, i.e.,

$$\lim_{h\to 0^+} \frac{\qquad}{\qquad} = \lim_{h\to 0^-} \frac{\qquad}{\qquad} = L < \infty$$

  or

$$\lim_{x\to a^+} \frac{\qquad}{\qquad} = \lim_{x\to a^-} \frac{\qquad}{\qquad} = L < \infty,$$

- f is differentiable at $a \Rightarrow f$ is continuous at a. (equivalently, not diff. $\Leftarrow$ not cont.)
  But $\Leftarrow$ is NOT necessarily true! Counterexample: $f(x) = |x|$ at $x = 0$.

- How to determine if f is differentiable at a?
  1. First check if f is defined at a. If not, then f is not differentiable at a.
  2. If f is defined at a, then check if f is continuous at a, i.e., is $f(a) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$? If not, then f is not differentiable at a.
  3. If f is continuous at a, then we need to check the left and the right derivatives: $\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} = L < \infty$.

  * Caution: Do not use $\lim_{x\to a^+} f'(x) = \lim_{x\to a^-} f'(x)$, i.e., do not take derivatives with rules and then plug in a.

  * Caution: If it's a break point of a piecewise defined funtion, be careful about which branch to use for evaluating $f(a)$ (see DQ3).

Sec.3.1–Sec.3.6 Most of the contents are derivative tools that were in the previous review worksheet. Here I only put some miscellaneous facts. Also see LQ2.

- **Differentiation Rules**: Please see the worksheet given at 10/15,
  or "F19_MATH241073D101D_1015T_DiffRules_ sec3-1to3-6.pdf" on Discussion Canvas.

- General equation for the tangent line at $(a, f(a))$: ____________________

- General equation for the normal line at $(a, f(a))$: ____________________

- **Two limits** from 3.3. Do examples (e.g., #39–50 from the textbook).

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \underline{\qquad\qquad} \qquad\qquad \lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = \underline{\qquad\qquad}$$

  Examples:

$$\lim_{t\to 0}\frac{\sin t}{t+\tan t} \qquad\qquad \lim_{\theta\to 0}\frac{\cos\theta-1}{2\theta^2} \qquad\qquad \lim_{x\to 0}\frac{\cos x - 1}{\sin x}$$

- e **as a limit** from 3.6

$$e = \lim_{x\to 0}\underline{\qquad\qquad\qquad} = \lim_{n\to\infty}\underline{\qquad\qquad\qquad}$$

Sec 4.1

- Definition of a critical number c of f (there're two cases): ________________
  *Note that c must be in the domain of f. Otherwise, say $f = 1/x$, $f' = -1/x^2$, then $x = 0$ is not a critical number.

- **Fermat's Theorem**: f has a local extremum at $c \Rightarrow c$ is a critical number of f.
  But $\Leftarrow$ is NOT necessarily true! Counterexample: $f(x) = x^3$ at $x = 0$.

- **The Closed Interval Method**
  Example: Given $f(x)$ and an interval $[a, b]$, say $f(x) = x^3 - 3x^2 + 1$, $[-\frac{1}{2}, 4]$, find the absolute maximum and minimum values of f in the given interval. (f is continuous on $[a, b]$)
  *The possible candidates are critical nubmers (when $f'(x) = 0$ or DNE), or the endpoints a, b.

  1. Take derivatives of f:
     $f'(x) = 3x^2 - 6x$
  2. Find critical numbers, i.e., solve for $f'(x) = 0$ or DNE:
     $3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0 \Rightarrow x = 0, 2$ *Check if they're in the given interval.
  3. Plug in the critical numbers and the endpoints to the *original function* f:
     $f(0) = 0^3 - 3\cdot 0^2 + 1 = 1$, $f(2) = 2^3 - 3\cdot 2^2 + 1 = 8 - 12 + 1 = -3$,
     $f(-1/2) = -\frac{1}{8} - \frac{3}{4} + 1 = \frac{1}{8}$, $f(4) = 64 - 48 + 1 = 17$.
  4. The largest number in step 3 is the absolute maximum, while the smallest is the absolute minimum:
     Ans: f has absolute maximum at 4, abs. max. value $f(4) = 17$; f has abs. min. at 2, abs. min. value $f(2) = -3$.

Sec.4.2

Four theorems: Here I give the theorems in short. Please try to rewrite/state in your own words. Also see the table in the worksheet given at 10/17. Pictures might be helpful. Those with names are the most important, i.e., Rolle's and MVT.

Symbols: $\forall$=“For all”, $\exists$=“exists”, $\Rightarrow$=“then”, $\in$=“in”, s.t.=“such that”.

- **Rolle's Theorem:**
  f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b) \Rightarrow \exists c \in (a, b)$ s.t. $f'(c) = 0$.

- **The Mean Value Theorem:**
  f is continuous on $[a, b]$, differentiable on $(a, b) \Rightarrow \exists c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \text{ or } (b-a)f'(c) = f(b) - f(a).$$

- **Theorem 5:** $\forall x \in (a,b), \quad f'(x) = 0 \Rightarrow f$ is constant on (a,b).
- **Corollary 7:** $\forall x \in (a,b), \quad f'(x) = g'(x) \Rightarrow f - g$ is constant on (a,b), i.e., $f(x) = g(x) + c$.

Sec 4.3
Relationships between f and f':

- $f' > 0$:
- $f' < 0$:
- $f' = 0$:

Relationships between f and f'':

- $f'' > 0$:
- $f'' < 0$:
- $f'' = 0$:

Two tests for local maximum/minimum:

- **The First Derivative Test**:
  Suppose c is a critical number, and f is continuous
  1. If __________________________________, then f has a local maximum at c.
  2. If __________________________________, then f has a local minimum at c.
  3. If there's no sign changes of f' at c, then f has no local extremum at c.
- **The Second Derivative Test**:
  Suppose f'' is continuous near c.
  1. If ____________, and ____________, then f has a local maximum at c.
  2. If ____________, and ____________, then f has a local minimum at c.

Remarks: Here are a few general questions you can ask yourself. Even better to explain to your classmate(s). "*If you can't explain clearly, then you don't really understand.* "–my Fluid Dynamics professor.

- About definitions:
  - Can you state the definition clearly in math symbols and terminologies?
  - Can you explain to anyone what is ______ in simple words or graphs?
- About theorems/"tests":
  - Can you state the theorem clearly? What are the hypotheses? What's the conclusion(s)?
  - Can you come up with a counterexample that the theorem fails, when any of the hypotheses is not satisfied?
  - Is the converse of the theorem true? If not, can you come up with a counterexample?