Calculus(II) 0412 Inclass Homework 9

Inclass 1

Use the method of Lagrange multipliers to find the minimum value of

$$f(x,y) = x^2 + (y-2)^2 \qquad \text{on the hyperbola} \qquad x^2 - y^2 = 1.$$

sol.

First, we need to solve the system of $\nabla f = \lambda \nabla g$ and $g(x,y)$:

$$\begin{cases} f_x = \lambda g_x : 2x = 2x\lambda \Rightarrow 2x(\lambda - 1) = 0 \ldots\ldots (1) \\ f_y = \lambda g_y : 2(y-2) = -2y\lambda \Rightarrow 2y(1+\lambda) = 4 \ldots\ldots (2) \\ x^2 - y^2 = 1 \ldots\ldots (3) \end{cases}$$

(1) gives two cases, $x = 0$ or $\lambda = 1$:

- If $x = 0$, then (3) gives
$$y^2 = -1,$$
which has no real solution for y. Thus, $x \neq 0$.
- If $\lambda = 1$, then (2) gives
$$4y = 4 \Rightarrow y = 1,$$
and therefore (3) gives
$$x^2 = 2 \Rightarrow x = \pm\sqrt{2}.$$

Therefore, we have two solutions for the system:

$$(x,y,\lambda) = (\sqrt{2}, 1, 1) \text{ and } (x,y,\lambda) = (-\sqrt{2}, 1, 1).$$

Substituting two points $(x,y) = (\sqrt{2}, 1)$ and $(x,y) = (-\sqrt{2}, 1)$ into f yields

$$f(\sqrt{2},1) = (\sqrt{2})^2 + (1-2)^2 = 3 \qquad \text{and} \qquad f(-\sqrt{2},1) = (-\sqrt{2})^2 + (1-2)^2 = 3.$$

Hence, the minimum value of f on g is $f(\pm\sqrt{2}, 1) = 3$.

Note:

- We need to find ALL solutions of the system of $\nabla f = \lambda \nabla g$ and $g(x,y)$ to make sure that we really get the minimum value.
- Some students might get confuse when solving the system of $\nabla f = \lambda \nabla g$ and $g(x,y)$. You can just view λ as another variable that you find comfortable with, say z. Then it is actually a high school problem.
- Some students have shown that another point on g gives greater value then $(\pm\sqrt{2}, 1)$ and concluded by this that $f(\pm\sqrt{2}, 1) = 3$ is really the minimum value. This is WRONG since we need to check the whole neighbourhood rather than just an arbitrary point.
In fact, given the existence of the extreme values, the method of Lagrange multipliers already guarantees that the values of f on the solutions of the system must contain the extreme values. So here, it is okay not to prove the minimality if you already check all solutions of the system.

Inclass 2

Consider the function
$$f(x,y) = 2x^2 + y^2 - xy - 7y.$$
Show that $(1,4)$ is a critical point and compute $f(1+h, 4+k)$.

Use this result and the definition of a local minimum to show that $(1,4)$ is a local minimum.

sol.

(i) According to the definition of critical point on p. 946, we need to check whether $f_x(1,4) = 0$ and $f_y(1,4) = 0$. Substituting $(x,y) = (1,4)$ into
$$f_x(x,y) = 4x - y \qquad \text{and} \qquad f_y(x,y) = 2y - x - 7$$
yields
$$f_x(1,4) = 4 - 4 = 0 \qquad \text{and} \qquad f_y(1,4) = 8 - 1 - 7 = 0.$$
Therefore, $(1,4)$ is a critical point.

(ii)
$$\begin{aligned} f(1+h, 4+k) &= 2(1 + 2h + h^2) + 16 + 8k + k^2 - 4 - 4h - k - kh - 28 - 7k \\ &= 2h^2 - kh + k^2 - 14. \end{aligned}$$
The definition of local minimum is on p. 946:

If $\mathbf{f(x,y) \geq f(a,b)}$ *when* (x,y) *is near* (a,b)*, then* f *has a local minimum at* (a,b) *and* $f(a,b)$ *is a local minimum value.*

So, we need to show that
$$f(1+h, 4+k) - f(1,4) \geq 0.$$
Note that
$$f(1,4) = 2 + 16 - 4 - 28 = -14.$$
Hence,
$$\begin{aligned} f(1+h, 4+k) - f(1,4) &= 2h^2 - kh + k^2 \\ &= 2\left(h^2 - \frac{1}{2}kh + \frac{k^2}{16}\right) + k^2 - 2 \cdot \frac{k^2}{16} \\ &= 2\left(h - \frac{k}{4}\right)^2 + \frac{7}{8}k^2 \geq 0 \end{aligned}$$
for any $h, k \in \mathbb{R}$. Therefore, $(1,4)$ is a local minimum.

Note:

- Don't confuse "$f(x,y) \geq f(a,b)$ *when* (x,y) *is near* (a,b)" with "$\lim_{(x,y)\to(a,b)} f(x,y) \geq f(a,b)$". Even if $\lim_{(x,y)\to(a,b)} f(x,y) \geq f(a,b)$, the values near (a,b) can be anything, not necessarily $\geq f(a,b)$.