Calculus(II) 0412 Homework 8 Part II

Online 3

Let $x = s + t$ and $y = s - t$. Show that for any differentiable function $f(x, y)$, we have

$$\left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2 = \frac{\partial f}{\partial s}\frac{\partial f}{\partial t}.$$

sol.

Applying the chain rule of case 2 in p. 926, we have

$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\\ &= \frac{\partial f}{\partial x}\frac{\partial}{\partial s}(s+t) + \frac{\partial f}{\partial y}\frac{\partial}{\partial s}(s-t)\\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\\ &= \frac{\partial f}{\partial x}\frac{\partial}{\partial t}(s+t) + \frac{\partial f}{\partial y}\frac{\partial}{\partial t}(s-t)\\ &= \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\end{aligned}$$

Therefore,

$$\frac{\partial f}{\partial s}\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) = \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2$$

as claimed.

Online 4

Let $f(x, y, z) = F(r)$ with $r = \sqrt{x^2 + y^2 + z^2}$. Then,

$$\nabla f = F'(r)\frac{\mathbf{r}}{|\mathbf{r}|},$$

where $\mathbf{r} = \langle x, y, z\rangle$.

sol.

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Clearly, $\frac{\partial f}{\partial x} = \frac{\partial F}{\partial x}, \frac{\partial f}{\partial y} = \frac{\partial F}{\partial y}$ and $\frac{\partial f}{\partial y} = \frac{\partial F}{\partial y}$.
$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$ can derive from the chain rule of general version (p. 927) that

F is a differentiable function of 1 variable r,
and r is a differentiable function of 3 variables x, y, z

Hence, we have

$$\frac{\partial F}{\partial x} = \frac{dF}{dr}\frac{\partial r}{\partial x}, \qquad \frac{\partial F}{\partial y} = \frac{dF}{dr}\frac{\partial r}{\partial y} \qquad \text{and} \qquad \frac{\partial F}{\partial z} = \frac{dF}{dr}\frac{\partial r}{\partial z}.$$

Thus,

$$\begin{aligned}\nabla f &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \frac{dF}{dr}\left\langle \frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z} \right\rangle \\ &= \frac{dF}{dr}\left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle \\ &= \frac{F'(r)}{r}\langle x, y, z\rangle = F'(r)\frac{\mathbf{r}}{|\mathbf{r}|}.\end{aligned}$$

Note:

- Only when the function f is of single variable can we use the notation $\frac{df}{dx}$ or f'. Otherwise we should use the partial derivative $\frac{\partial f}{\partial x}$ or f_x.
- If you work from the right-hand-side, note that

$$F'(r) \neq \frac{\partial F}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial r}$$

because all of x, y and z are NOT a function of the single variable r.

Online 5

Suppose $f(x,y)$ is differentiable at (a,b) with $\nabla f(a,b) \neq \mathbf{0}$. Find the directional derivative of $f(x,y)$ in the direction of

$$\frac{\partial f}{\partial y}(a,b)\mathbf{i} - \frac{\partial f}{\partial x}(a,b)\mathbf{j}$$

Also, give a geometric interpretation of your result.

sol.

Let $\mathbf{v} = \frac{\partial f}{\partial y}(a,b)\mathbf{i} - \frac{\partial f}{\partial x}(a,b)\mathbf{j}$ and $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$ be the unit vector.

Then, by equation 9 of p. 936, we have

$$\begin{aligned}
D_{\mathbf{u}}f(x,y) &= \nabla f(x,y) \cdot \mathbf{u} \\
&= \left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right\rangle \cdot \frac{\left\langle \frac{\partial f}{\partial y}(a,b), -\frac{\partial f}{\partial x}(a,b) \right\rangle}{\sqrt{f_y^2(a,b) + f_x^2(a,b)}} \\
&= \frac{1}{\sqrt{f_y^2(a,b) + f_x^2(a,b)}} \left(\frac{\partial f}{\partial x}(x,y)\frac{\partial f}{\partial y}(a,b) - \frac{\partial f}{\partial y}(x,y)\frac{\partial f}{\partial x}(a,b) \right)
\end{aligned}$$

At the point (a,b),

$$\begin{aligned}
D_{\mathbf{u}}f(a,b) &= \frac{1}{\sqrt{f_y^2(a,b) + f_x^2(a,b)}} \left(\frac{\partial f}{\partial x}(a,b)\frac{\partial f}{\partial y}(a,b) - \frac{\partial f}{\partial y}(a,b)\frac{\partial f}{\partial x}(a,b) \right) \\
&= 0
\end{aligned}$$

Clearly, $\mathbf{u}$ is perpendicular to $\nabla f(a,b)$. As shown in Figure 11 in p. 942, the gradient vector at (a,b), $\nabla f(a,b)$, must be perpendicular to the level curve that passes through (a,b). This implies that $\mathbf{u}$ is the direction of the level curve. Since the function is constant on level curves, the directional derivative, which means the rate of change, must be 0 in the direction of $\mathbf{u}$.

Note:

- Many students only use the concept given in Section 12.3 that "Two vectors are orthogonal if and only if their dot product is 0." Here in Section 14.6, we need to consider that $D_{\mathbf{u}}f(x,y)$ is the directional derivative rather than just a dot product.