Calculus(II) 0412 Homework 7 Part I

Textbook 14.2.25

Find $h(x, y) = g(f(x, y))$ and the set on which h is continuous.

$$g(t) = t^2 + \sqrt{t}, \quad f(x, y) = 2x + 3y - 6$$

sol.

$$h(x, y) = g(2x + 3y - 6) = (2x + 3y - 6)^2 + \sqrt{2x + 3y - 6}.$$

Since $f(x)$ is a polynomial , it is continuous everywhere on $\mathbb{R}^2$. And $g(t)$ is continuous on $\{t : t \geq 0\}$ because of the square root. Hence, $h(x, y)$ is continuous on

$$\{(x, y) \in \mathbb{R}^2 : 2x + 2y - 6 \geq 0\} = \left\{(x, y) \in \mathbb{R}^2 : y \geq -\frac{2}{3}x + 2\right\}.$$

Note:

- $g(t)$ and $h(x, y)$ are NOT polynomial.

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Determine **the set of points** at which the function is continuous.

$$f(x, y) = \begin{cases} \frac{x^2y^3}{2x^2+y^2} & \text{if } (x, y) \neq (0, 0); \\ 1 \quad \text{if } (x, y) = (0, 0). \end{cases}$$

sol.

When $(x, y) \neq (0, 0)$, $f(x)$ is a rational function which is continuous on its domain $\mathbb{R}^2 \setminus (0, 0)$. We need to check whether it is continuous on the origin $(0, 0)$, i. e. $\lim_{(x,y)\to(0,0)} f(x, y) \stackrel{?}{=} f(0, 0) = 1$. When approaching $(0, 0)$ along the x-axis, we have

$$f(x, 0) = \frac{x^2 \cdot 0}{2x^2 + 0} = 0 \qquad \text{for all } x \neq 0.$$

This implies that the limit $\lim_{(x,y)\to(0,0)} f(x, y)$, if exists, is impossible to equal to 1. Therefore $f(x, y)$ is discontinuous on $(0, 0)$ and the set of points at which $f(x, y)$ continuous is $\mathbb{R}^2 \setminus (0, 0)$.

Textbook 14.3.61

$$\begin{aligned} u &= \cos(x^2y) \\ u_x &= -\sin(x^2y) \cdot 2xy; \qquad u_y = -\sin(x^2y) \cdot x^2 \\ u_{xy} &= -\cos(x^2y)2xy \cdot x^2 - \sin(x^2y) \cdot 2x \\ u_{yx} &= -\cos(x^2y)x^2 \cdot 2xy - \sin(x^2y) \cdot 2x \\ &\Rightarrow u_{xy} = u_{yx}. \end{aligned}$$

Online 1

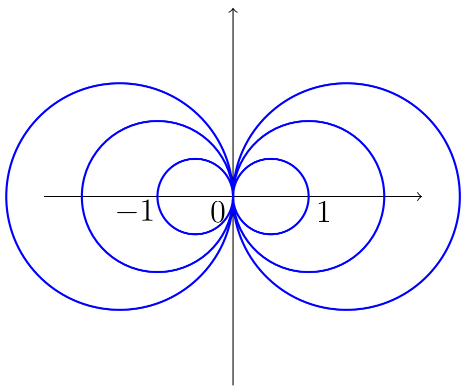
Make a sketch of the level curves of the following function

$$f(x, y) = \frac{x}{x^2 + y^2}.$$

sol.

$$\begin{aligned} f(x, y) =& \frac{x}{x^2+y^2} = k \\ \Rightarrow& kx^2 - x + ky^2 = 0 \\ \Rightarrow& k\left(x^2 - \frac{x}{k} + \left(\frac{1}{2k}\right)^2 - \left(\frac{1}{2k}\right)^2\right) + ky = 0 \\ \Rightarrow& \left(x - \frac{1}{2k}\right) + y = \left(\frac{1}{2k}\right)^2 \end{aligned}$$

The level curves are the circles with the center at $(\frac{1}{2k}, 0)$ and the radius $\left|\frac{1}{2k}\right|$.



Online 2

$$f(x, y) = \frac{x^3 + y^3}{x^2 + y^2}$$

(a) Show that for all $(x, y) \neq (0, 0)$, $|f(x, y)| \leq |x| + |y|$:

$$\left|\frac{x^3+y^3}{x^2+y^2}\right| \leq \left|\frac{x^3 + x^2y + xy^2 + y^3}{x^2+y^2}\right| = \left|\frac{(x+y)(x^2+y^2)}{x^2+y^2}\right| = |x+y| \leq |x| + |y|.$$

The last equality holds when x and y are of the same sign.

(b) Use part (a) and the precise definition of the limit to show that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$:

As shown in Definition 1 of Section 14.2, we need to show that

$$\begin{aligned} &\forall \epsilon > 0 \exists \delta > 0 \text{ such that} \\ &(x, y) \in D \text{ and } 0 < \sqrt{x^2+y^2} < \delta \Rightarrow |f(x, y) - 0| < \epsilon, \end{aligned}$$

where $D = \mathbb{R}^2 \setminus (0,0)$ is the domain of f. By (a) and the triangle inequality, we know that

$$|f(x,y)| \leq |x| + |y| \leq 2\sqrt{|x|^2 + |y|^2} = 2\sqrt{x^2 + y^2} < 2\delta.$$

Thus, for a given ϵ, we can choose $\delta = \frac{\epsilon}{2}$ such that

$$|f(x,y)| \leq 2\delta = \epsilon.$$

By definition, this implies that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Note:

- δ cannot be a function of x or y.