Calculus(II) 0412 Homework 5 Part II

Online 3

Let $\mathbf{a} \neq \mathbf{0}$. Show that $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ and $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ implies that $\mathbf{b} = \mathbf{c}$.

sol.

(i) From No. 4 of Theorem 11 of Section 12.4 (p. 812), we know that

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \Longrightarrow \mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}.$$

Let θ be the angle between $\mathbf{a}$ and $\mathbf{b} - \mathbf{c}$ such that $0 \leq \theta \leq \pi$.
Then Theorem 9 (p. 810) gives

$$|\mathbf{a}||\mathbf{b} - \mathbf{c}| \sin\theta = 0.$$

If $\mathbf{b} - \mathbf{c} \neq 0$ (i.e. $\mathbf{b} \neq \mathbf{c}$), then the above equation implies that $\mathbf{a}$ and $\mathbf{b} - \mathbf{c}$ are parallel by Corollary 10 (p. 811).

(ii) From No. 3 of Theorem 2 of Section 12.3 (p. 801), we know that

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} \Longrightarrow \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0.$$

Then Theorem 3 (p. 801) gives

$$|\mathbf{a}||\mathbf{b} - \mathbf{c}| \cos\theta = 0.$$

This implies that $\mathbf{a}$ and $\mathbf{b} - \mathbf{c}$ are orthogonal by Corollary 7 (p. 803).

Clearly, it is impossible that two nonzero vectors are both parallel and orthogonal. Since $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} - \mathbf{c}$ must be $\mathbf{0}$. Thus, $\mathbf{b} = \mathbf{c}$.

Note:

- Some students apply Theorem 9 (p. 810) and Theorem 3 (p. 801) directly:

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \Longrightarrow |\mathbf{a}||\mathbf{b}| \sin\theta_1 = |\mathbf{a}||\mathbf{c}| \sin\theta_2, \tag{1}$$

and

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} \Longrightarrow |\mathbf{a}||\mathbf{b}| \cos\theta_1 = |\mathbf{a}||\mathbf{c}| \cos\theta_2, \tag{2}$$

where θ_1 (*resp.* θ_2) is the angle between $\mathbf{a}$ and $\mathbf{b}$ (*resp.* $\mathbf{c}$). $0 \leq \theta_1 \leq \pi$ and $0 \leq \theta_2 \leq \pi$. Then argue that (1) $\div$ (2): $\tan\theta_1 = \tan\theta_2 \Longrightarrow \theta_1 = \theta_2$. Be careful that you should first assume that $|\mathbf{b}|$, $|\mathbf{c}|$, $\cos\theta_1$ and $\cos\theta_2$ are nonzero before you divide (1) by (2). After that, you should discuss the case when those are zero since we are asking for a general case.

Online 4

Show that
$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$
for all vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$.

sol.

Let $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3),$ and $\mathbf{w} = (w_1, w_2, w_3)$.

Then,

$$\begin{aligned}
&\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \\
&\quad \mathbf{u} \times (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1) \\
&\quad = \Big(u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3), \\
&\qquad u_3(v_2w_3 - v_3w_2) - u_1(v_1w_2 - v_2w_1), \\
&\qquad u_1(v_3w_1 - v_1w_3) - u_2(v_2w_3 - v_3w_2)\Big) \\
&\quad = \Big((u_2w_2 + u_3w_3)v_1 - (u_2v_2 + u_3v_3)w_1 - u_1v_1w_1 + u_1v_1w_1, \\
&\qquad (u_1w_1 + u_3w_3)v_2 - (u_1v_1 + u_3v_3)w_2 - u_2v_2w_2 + u_2v_2w_2, \\
&\qquad (u_1w_1 + u_2w_2)v_3 - (u_1v_1 + u_2v_2)w_3 - u_3v_3w_3 + u_3v_3w_3\Big) \\
&\quad = \Big((u_1w_1 + u_2w_2 + u_3w_3)v_1 - (u_1v_1 + u_2v_2 + u_3v_3)w_1, \\
&\qquad (u_1w_1 + u_2w_2 + u_3w_3)v_2 - (u_1v_1 + u_2v_2 + u_3v_3)w_2, \\
&\qquad (u_1w_1 + u_2w_2 + u_3w_3)v_3 - (u_1v_1 + u_2v_2 + u_3v_3)w_3\Big) \\
&\quad = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.
\end{aligned}$$

<u>Online 5</u>

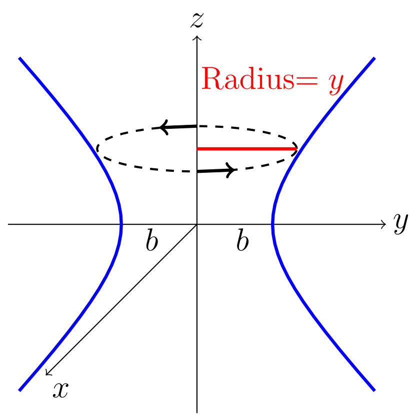
The hyperbola $c^2y^2 - b^2z^2 - b^2c^2 = 0$ is revolved about the z-axis. Find an equation for the resulting surface. Is it a quadratic surface? If yes, which one; if no, why not?

<u>sol.</u>

Divide $c^2y^2 - b^2z^2 - b^2c^2 = 0$ by b^2c^2:

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

This is a hyperbola according to p. 674.



When we revolve the hyperbola about the z-axis, the horizontal traces (parallel to the xy-plane) are circles:

$$x^2 + y^2 = r^2.$$

Moreover, the radius r is determine by the y component on the hyperbola:

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$
$$\Longrightarrow y^2 = b^2 + \frac{b^2z^2}{c^2}.$$

Therefore, the equation of the resulting surface is

$$x^2 + y^2 = r^2 = b^2 + \frac{b^2z^2}{c^2} \Longrightarrow \frac{x^2}{b^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{3}$$

By comparing (3) with the equations on Table 1 of Section 12.6 (p. 830), we conclude that this is a **hyperboloid of one sheet**.