Calculus(II) 0412 Homework 4 Part I

Textbook 11.10.35

Use a Maclaurin series in Table 1 to obtain the Maclaurin series for the given function.

$$\frac{x}{\sqrt{4+x^2}}$$

$$\frac{x}{\sqrt{4+x^2}} = \frac{x}{\sqrt{4\left(1+\frac{x^2}{4}\right)}} = \frac{x}{2}\left(1+\frac{x^2}{4}\right)^{-\frac{1}{2}}$$

We use the Binomial Series expansion (p. 761)

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + ... \qquad k\in\mathbb{R} \text{ and } |x|<1.$$

Now $k=-\frac{1}{2}$ and when $|\frac{x^2}{4}|<1$:

$$\begin{aligned}
\frac{x}{2}\left(1+\frac{x^2}{4}\right)^{-\frac{1}{2}} &= \frac{x}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x^2}{4}\right)^n \\
&= \frac{x}{2}\left(1+\left(-\frac{1}{2}\right)\frac{x^2}{4} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x^2}{4}\right)^2 + ...\right) \\
&= \frac{x}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!}\left(\frac{x^2}{4}\right)^n \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^{3n+1}n!}x^{2n+1}
\end{aligned}$$

Note:

- $(2n-1)!! = 1\cdot 3\cdot 5\cdot ...\cdot 2n-1$.
- This exercise asks for Maclaurin series, which is a power series representation at 0. So your answer should be of the form (p. 754)

$$f(x) = \sum_{n=0}^{\infty} c_n x^n.$$

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Find the sum of the series

$$\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{4}\right)^{2n+1} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Note:

- $\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$.

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$$E = \frac{q}{D^2} - \frac{q}{(D+d)^2}$$

By expending E as a series in powers of $\frac{d}{D}$, show that E is approximately proportional to $\frac{1}{D^3}$ when P is far away from the dipole.
According to the picture shown in p. 775, "P is far away from the dipole" means that $D \gg d$.

$$\begin{aligned}
E &= \frac{q}{D^2} - \frac{q}{(D+d)^2} = \frac{q}{D^2} - \frac{q}{D^2\left(1+\frac{d}{D}\right)^2} \\
&= \frac{q}{D^2}\left(1 - \left(1+\frac{d}{D}\right)^{-2}\right) \\
&= \frac{q}{D^2}\left(1 - \sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{d}{D}\right)^n\right) \\
&= \frac{q}{D^2}\left(1 - \left(1 + (-2)\left(\frac{d}{D}\right) + \frac{(-2)(-3)}{2!}\left(\frac{d}{D}\right)^2 + \dots\right)\right) \\
&= \frac{q}{D^2}\left(2\left(\frac{d}{D}\right) - 3\left(\frac{d}{D}\right)^2 + \dots\right)
\end{aligned}$$

When $D \gg d$, $1 \gg \frac{d}{D}$. So $\frac{d}{D} \gg \left(\frac{d}{D}\right)^2 \gg \dots$ and therefore

$$E \approx \frac{2qd}{D^3} \propto \frac{1}{D^3}.$$

Online 1

$$e^x \stackrel{?}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Let $g(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(i) Show that $g'(x) = g(x)$ for $x \in \mathbb{R}$.
First show that $g(x)$ converges for $x \in \mathbb{R}$:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 < 1$$

So $g(x)$ is differentiable on $\mathbb{R}$:

$$\begin{aligned}
g'(x) &= \sum_{n=1}^{\infty} n \cdot \frac{1}{n!} \cdot x^{n-1} \\
&= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = g(x).
\end{aligned}$$

(ii) Define $h(x) = e^{-x}g(x)$.

$$\begin{aligned} h'(x) &= \big(e^{-x}g(x)\big)' = (e^{-x})'g(x) + e^{-x}g'(x) \\ &= -e^{-x}g(x) + e^{-x}g(x) && \text{By (i), } g'(x) = g(x) \\ &= 0. \end{aligned}$$

Since $h'(x) = 0$, we know that $h(x) = c$ is a constant. At $x = 0$,

$$h(0) = c = e^0 g(0) = 1 \cdot \sum_{n=0}^{\infty} \frac{0^n}{n!} = 1 \cdot 1 = 1.$$

Therefore, $g(x) = ce^x = e^x$.

Note:

- See p.748, Theorem 2 for the term-by-term differentiation and integration. You would not lose any point if you didn't check the convergence since it's already proved in the lecture. But it's good to keep in mind that a power series $\sum c_n(x-a)^n$ is only differentiable on the interval $(a-R, a+R)$, where $R > 0$ is the radius of convergence.
- Since our goal is to prove that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we cannot use this equation in our proof.

Online 2

(i) The Taylor series of the function f at a (p. 754) is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Now $f(x) = e^x$, $a = 3$ and $f^{(n)}(x) = e^x$ for all n:

$$e^x = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n.$$

Apply the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x-3}{n+1}\right| = 0 < 1$$

So the series converges for all $x \in \mathbb{R}$.

(ii) Write e^x as

$$e^x = T_n(x) + R_n(x) = \sum_{i=0}^{n} \frac{e^3}{i!}(x-3)^i + R_n(x).$$

The Taylor's inequality (p. 756):
If $|f^{n+1}(x)| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \textit{for } |x-a| \le d.$$

Now $f(x) = e^x$, $a = 3$ and $f^{(n+1)}(x) = e^x$ for all n.
If d is any positive number and $|x-3| \le d$, then $-d+3 \le x \le d+3$ and

$$|f^{(n+1)}(x)| = e^{|x|} \le e^{d+3} = M.$$

So

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} = \frac{e^{d+3}}{(n+1)!}|x-3|^{n+1}$$

and

$$\begin{aligned}\lim_{n\to\infty} |R_n(x)| &\le \lim_{n\to\infty} \frac{e^{d+3}}{(n+1)!}|x-3|^{n+1} \\ &= e^{d+3} \lim_{n\to\infty} \frac{|x-3|^{n+1}}{(n+1)!} = 0\end{aligned}$$

It follows from the Squeeze Theorem (the left side is $0 \le |R_n(x)|$) that $\lim_{n\to\infty} |R_n(x)| = 0$ and therefore $\lim_{n\to\infty} R_n(x) = 0$ for all x. By Theorem 8 (p. 756), we obtain

$$e^x = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n \qquad \text{for all } x.$$

Note:

- See p.754-757.
- Don't confuse with the well-known series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, which is the Taylor series of e^x at $x = 0$.
- The Taylor's inequality needs a bound M that is regardless of x for $|f^{(n+1)}(x)|$.