Calculus(II) 0412 Inclass Homework 3

Inclass 1

Find radius and interval of convergence of the two power series

(1)

$$\sum_{n\geq 1} \frac{(-1)^n}{n^2}(x+2)^n$$

We use the ratio test with $a_n = \frac{(-1)^n}{n^2}(x+2)^n$:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{(n+1)^2}(x+2)^{n+1}}{\frac{(-1)^n}{n^2}(x+2)^n}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2|x+2| = |x+2|$$

The series converges if $|x+2|<1$. The endpoints:

- When $x=-3$
$$\sum_{n\geq 1}\frac{(-1)^{2n}}{n^2} = \sum_{n\geq 1}\frac{1}{n^2}$$
is a convergent p-series.
- When $x=-1$
$$\sum_{n\geq 1}\frac{(-1)^n}{n^2}$$
converges since $\sum_{n\geq 1}\left|\frac{(-1)^n}{n^2}\right| = \sum_{n\geq 1}\frac{1}{n^2}$ is a convergent p-series.

Therefore the radius of convergence is $R=1$ and the interval of convergence is $[-3,-1]$.

(2)

$$\sum_{n\geq 1} \frac{(-1)^n}{n^n}(x+2)^n$$

We use the root test with $a_n = \frac{(-1)^n}{n^n}(x+2)^n$:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{(-1)^n}{n^n}(x+2)^n\right|} = \lim_{n\to\infty}\frac{|x+2|}{n} = 0 < 1.$$

Therefore the series always converges and the radius of convergence is $R=\infty$ and the interval of convergence is $\mathbb{R}=(-\infty,\infty)$.

Note:

- See p.734-736.

Inclass 2

(a) Given:

- a_n converges to 0
- $\sum_{n\geq 1} b_n$ is absolute convergent.

a_n converges to 0 implies that $\forall\epsilon > 0 \quad \exists N$ such that

$$|a_n - 0| < \epsilon \quad \text{whenever } n > N.$$

We can choose $\epsilon = 1$ then $\exists N$ such that if $n > N$

$$|a_n| < 1$$
$$|a_n b_n| < |b_n|$$

Since $\sum_{n\geq 1} |b_n|$ converges, $\sum_{n\geq 1} |a_n b_n|$ also converges by comparison test and therefore $\sum_{n\geq 1} a_n b_n$ also converges.

(b) If we only given "$\sum_{n\geq 1} b_n$ is convergent" instead of "$\sum_{n\geq 1} b_n$ is absolute convergent", there is a counter example: Consider

$$a_n = \sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n}} \qquad b_n = \sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n}}$$

By the Alternating Series Test,

(i) $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ for all n

(ii) $\frac{1}{n} \to 0$ as $n \to \infty$

Therefore both $\sum a_n$ and $\sum b_n$ converges.
However,

$$\sum_{n\geq 1} a_n b_n = \sum_{n\geq 1} \frac{1}{n}$$

diverges (harmonic series).

Note:

- The ratio (root) test cannot reverse. A series $\sum a_n$ converges does NOT implies that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ nor $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists and of course NOT implies that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ nor $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.
- You can also apply #3 of the last homework, but need to be careful that the Limit Comparison Test is valid only when both a_n and b_n are POSITIVE. So the test should be done under the absolute values:

$$\lim_{n\to\infty} \frac{|a_n b_n|}{|b_n|} = \lim_{n\to\infty} |a_n| = 0.$$

Since $\sum_{n\geq 1} |b_n|$ converges, $\sum_{n\geq 1} |a_n b_n|$ also converges by #3-(a) of Homework 3 and therefore $\sum_{n\geq 1} a_n b_n$ also converges.