Calculus(II) 0412 Homework 2 Part II

Online 3

Given:

- a_n, $b_n > 0$
- $\lim_{n\to\infty} a_n/b_n = 0$

(a) $\lim_{n\to\infty} a_n/b_n = 0$ implies that $\forall\epsilon > 0 \quad \exists N \in \mathbb{N}$ such that

$$\left|\frac{a_n}{b_n} - 0\right| < \epsilon \text{ whenever } n > N.$$

Since b_n is positive, the above inequality gives $a_n < \epsilon b_n$(otherwise the "<" sign might reverse). If $\sum b_n$ converges, so does $\sum \epsilon b_n$. Thus $\sum a_n$ converges by part (i) of the Comparison Test.

(b) We use part (a) with

$$a_n = \frac{\log n}{n^2} \qquad b_n = \frac{1}{n^{3/2}}$$

and obtain

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\log n}{n^{1/2}} = \lim_{x\to\infty}\frac{\log x}{x^{1/2}} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty}\frac{2}{x^{1/2}} = 0.$$

Since $\sum b_n$ is a p-series (p.717) with $3/2 > 1$, it converges and therefore $\sum a_n$ also converges by (a).

Note:

- See p.724-725 and Exercises 40-41 of Sec. 11.4
- The Limit Comparison Test in p.724 is valid only when $\lim_{n\to\infty} a_n/b_n = c$, $c \in (0,\infty)$. This Exercise aims to deal with the case when $c = 0$. On the other hand, Exercises 41 in p.727 is the case when $c = \infty$ that both series diverge.
- In (b), we cannot applied the l'Hospital's rule directly since n is DISCRETE and $\log n, n^{1/2}$ are NOT differentiable at all!

Online 4

Given:

- $p_n \leq Cn\log n$ for some constant $C > 0$.

Clearly, $p_n \leq Cn\log n$ gives

$$\frac{1}{p_n} \geq \frac{1}{Cn\log n}.$$

Note that $\frac{1}{Cn\log n}$ is not defined when $n \leq 1$ due to the logarithm. Since finite terms do not contribute to the convergence, we may consider

$$\sum_{n=2}^{\infty}\frac{1}{p_n} \geq \sum_{n=2}^{\infty}\frac{1}{Cn\log n}.$$

Let $f(x) = \frac{1}{Cx\log x}$. To apply The Integral Test (p.716), first check that

(i) $f(x)$ is continuous on $[2, \infty)$

(ii) $f(x)$ is positive on $[2, \infty)$

(iii) $f(x)$ is decreasing on $[2, \infty)$ since

$$f'(x) = -\frac{\log x + 1}{Cx^2 \log^2 x} < 0.$$

So we use the Integral Test:

$$\int_2^\infty \frac{1}{Cx \log x} dx = \frac{1}{C} \lim_{t\to\infty} \int_2^t \frac{1}{x \log x} dx = \frac{1}{C} \lim_{t\to\infty} \log(\log x)\Big|_2^t = \infty.$$

Therefore $\sum_{n=2}^\infty \frac{1}{Cn \log n}$ diverges and by the Comparison Test, $\sum_{n=2}^\infty \frac{1}{p_n}$ also diverges.
Note:

- Again, we cannot integrate $\frac{1}{Cn \log n}$ directly since n is DISCRETE and $\frac{1}{Cn \log n}$ is NOT integrable at all!

<u>Online 5</u>
This is an alternating series that $b_n = \frac{1}{(2n)!}$ and starting from $n = 0$.
(a) Use the Alternating Series Test (p.727), $b_n > 0$ and :

(i) $b_1 = \frac{1}{2} \le 1 = b_0$; $b_{n+1} = \frac{1}{(2(n+1))!} = \frac{1}{(2n+2)\cdot(2n+1)\cdot(2n)!} \le \frac{1}{(2n)!} = b_n$ for all n.

(ii) $\lim_{n\to\infty} \frac{1}{(2n)!} = 0$

Thus the series is convergent.
(b) We apply the Alternating Series Estimation Theorem (p.730), where the assumptions are checked in (a). The error $|R_n| = |s - s_n| \le b_{n+1}$. When $n = 4$, $b_n = \frac{1}{(2\cdot 4)!} \approx 2.5 \times 10^{-5} > 5 \times 10^{-6}$, and $b_{n+1} = \frac{1}{10!} \approx 2.8 \times 10^{-7} < 5 \times 10^{-6}$. So our approximation can be $s_4 = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} \approx 0.540302$.
Note:

- The inequality $|R_n| = |s - s_n| \le b_{n+1}$ means:
  - $s = \sum_{i=0}^\infty (-1)^i b_i$ is the exact sum
  - $s_n = \sum_{i=0}^n (-1)^i b_i$ is the approximate sum
  - $|R_n|$ is the error term that we need it $< 5 \times 10^{-6}$.

- Since we want an approximation with an error less than $\le 5 \times 10^{-6}$, we should write the answer at least to 10^{-6}. In fact $|s - s_n| \le b_{n+1}$ implies that

  $$-b_{n+1} + s_n \le s \le b_{n+1} + s_n,$$

  in this case we have

  $$0.5403023 \le s \le 0.5403028.$$

  So if you write the answer as ≈ 0.54, the error is greater than 3×10^{-4}!