Calculus(II) 0412 Homework 1 Part I

Textbook

11.1.48.
Since $-1 \le \sin 2n \le 1$,
$$\frac{-1}{1+\sqrt{n}} \le \frac{\sin 2n}{1+\sqrt{n}} \le \frac{1}{1+\sqrt{n}}.$$
And
$$\lim_{n\to\infty} \frac{\pm 1}{1+\sqrt{n}} \to 0.$$
Therefore by the Squeeze Theorem, a_n is also converges to 0 as $n \to \infty$.

11.1.77.
$$a_n = \frac{n}{n^2+1}$$
Let
$$f(x) = \frac{x}{x^2+1}$$
be a continuous function on positive x. Then
$$f'(x) = \frac{1-x^2}{x^2+1}.$$
The denominator is always positive for real x, and the numerator is ≤ 0 for all $x \ge 1$. Hence $f(x)$ is decreasing and so does $\{a_n\}$.
To check whether a_n is bounded, first note that $a_n \ge 0$ for $n > 0$. So $\{a_n\}$ is bounded below by 0. Also, the fact that $\{a_n\}$ is decreasing implies that $a_1 = \frac{1}{2}$ is an upper bound of $\{a_n\}$.

11.2.32.
$$\sum_{n=1}^{\infty} \frac{1+3^n}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n.$$

- $\sum_{n=1}^{\infty} (\frac{1}{2})^n$: $|r| = \frac{1}{2} < 1$ converge
- $\sum_{n=1}^{\infty} (\frac{3}{2})^n$: $|r| = \frac{3}{2} \ge 1$ diverge

Therefore the series $\sum_{n=1}^{\infty} \frac{1+3^n}{2^n}$ diverges.

Online 1
Let $b_n = \frac{1}{n}$, then $\lim_{n\to\infty} b_n = 0$. Consider a continuous function $f(x) = a^x$. By Theorem 7 of the textbook, we obtain
$$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} a^{\frac{1}{n}} = f(0) = 1.$$

Online 2

(a) By the precise definition of limit,
$\lim_{n\to\infty} a_{2n} = a$ implies that

$$\forall\epsilon > 0 \exists N_1 \text{ such that } |a_{2n} - a| < \epsilon \text{ whenever } 2n > N_1,$$

and $\lim_{n\to\infty} a_{2n+1} = a$ implies that

$$\forall\epsilon > 0 \exists N_2 \text{ such that } |a_{2n+1} - a| < \epsilon \text{ whenever } 2n+1 > N_2.$$

Let $N = \max(N_1, N_2)$ and consider $n > N$.

- When $n = 2k$ is even,

$$\forall\epsilon > 0 \exists N_1 \text{ such that } |a_{2k} - a| < \epsilon \text{ whenever } 2k > N > N_1,$$

- When $n = 2k+1$ is odd,

$$\forall\epsilon > 0 \exists N_2 \text{ such that } |a_{2k+1} - a| < \epsilon \text{ whenever } 2k+1 > N > N_2.$$

Therefore we have

$$\forall\epsilon > 0 \exists N \text{ such that } |a_n - a| < \epsilon \text{ whenever } n > N.$$

(b) Again, $\lim_{n\to\infty} a_n = a$ implies that

$$\forall\epsilon > 0 \exists N \text{ such that } |a_n - a| < \epsilon \text{ whenever } n > N.$$

Clearly, $2n+1 > 2n > n > N$, so we have $\lim_{n\to\infty} a_{2n} = a$ and $\lim_{n\to\infty} a_{2n+1} = a$.