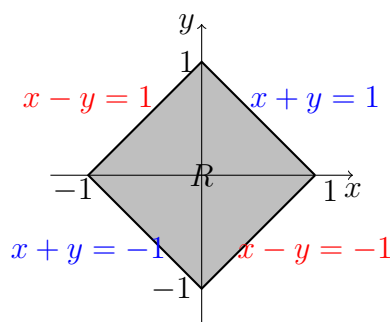# Calculus(II) 0412 Inclass Homework 12

Inclass 1

Use the change of variable $u = x + y$ and $v = x - y$ to compute the double integral

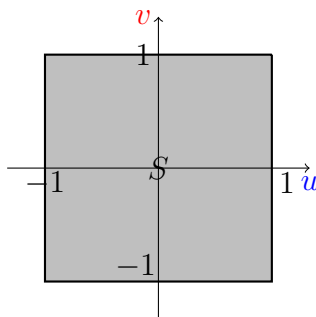$$\int\int_R (x+y)^2 e^{x^2-y^2}\,dxdy,$$

where R is the square with vertices $(1,0), (0,1), (-1,0)$ and $(0,-1)$.
  sol.

Draw the square R first:

So the image of $u = x + y$ and $v = x - y$ is

Also, $u = x + y$ and $v = x - y$ imply that $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$. Therefore, the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{vmatrix} = -\frac{1}{2}.$$
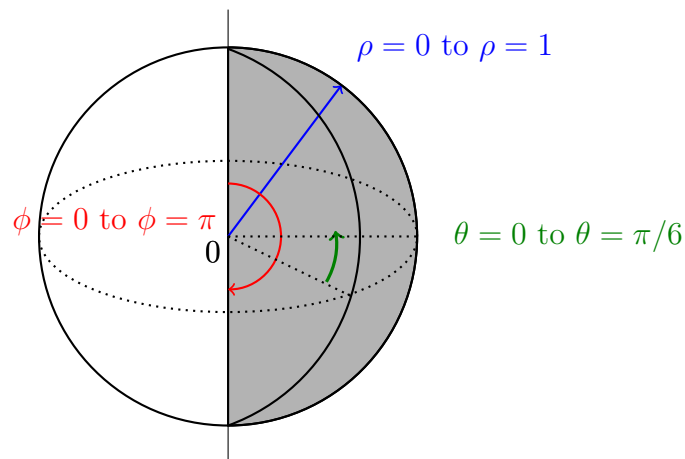
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Therefore, the integral is

$$
\begin{aligned}
\int\int_R (x+y)^2 e^{x^2-y^2}dxdy &= \int\int_S u^2 e^{uv}\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv\\
&= \frac{1}{2}\int_0^1 u^2\left(\int_0^1 e^{uv}dv\right)du\\
&= \frac{1}{2}\int_0^1 u^2\left[\frac{1}{u}e^{uv}\right]_0^1 du\\
&= \frac{1}{2}\int_0^1 ue^u - ue^{-u}du\\
\text{Intergration by parts}\quad &= \frac{1}{2}\left[ue^u - e^u + ue^{-u} + e^u\right]_0^1\\
&= 2e^{-1}.
\end{aligned}
$$

Inclass 2

Find he volume of the smaller wedge cut out from the unit sphere $x^2 + y^2 + z^2 = 1$ by two planes intersection at a diameter of the sphere and making an angle of $\pi/6$.
  sol.

Use the spherical coordinate, we have:

$$
\begin{aligned}
\int_0^\pi\int_0^{\frac{\pi}{6}}\int_0^1 \rho^2\sin\phi\, d\rho d\theta d\phi &= \int_0^\pi\int_0^{\frac{\pi}{6}}\sin\phi\cdot\frac{1}{3}d\theta d\phi\\
&= \frac{1}{3}\frac{\pi}{6}\int_0^\pi\sin\phi\cdot d\phi\\
&= \frac{\pi}{18}[-\cos\phi]_0^\pi = \frac{\pi}{9}.
\end{aligned}
$$