

Calculus(II) 0412 Homework 11 Part II

Online 3

Find the volume of the solid in the first octant which is bounded by $z = x$, $y - x = 2$ and $y = x^2$. Where is the center of mass if the mass density is constant?

sol.

The volume is given by the integral:

$$\begin{aligned} \int_0^2 \int_{x^2}^{x+2} \int_0^x dz dy dx &= \int_0^2 \int_{x^2}^{x+2} x dy dx = \int_0^2 x^2 + 2x - x^3 dx \\ &= \left[\frac{1}{3}x^3 + x^2 - \frac{1}{4}x^4 \right]_0^2 = \frac{8}{3}. \end{aligned}$$

Let ρ be the constant mass density so that the mass $M = \frac{8}{3}\rho$. According to p. 1024, the center of mass is given by

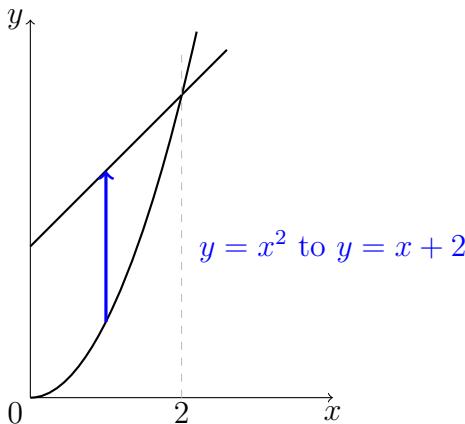
$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right),$$

where

$$\begin{aligned} M_{yz} &= \int_0^2 \int_{x^2}^{x+2} \int_0^x x \rho dz dy dx = \rho \int_0^2 \int_{x^2}^{x+2} x^2 dy dx = \rho \int_0^2 x^3 + 2x^2 - x^4 dx \\ &= \rho \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^5}{5} \right]_0^2 = \rho \left(4 + \frac{16}{3} - \frac{32}{5} \right) = \frac{44}{15}\rho, \\ M_{xz} &= \int_0^2 \int_{x^2}^{x+2} \int_0^x y \rho dz dy dx = \rho \int_0^2 \int_{x^2}^{x+2} xy dy dx = \rho \int_0^2 \frac{x}{2}(x^2 + 4x + 4) - \frac{x^5}{2} dx \\ &= \rho \left[\frac{x^4}{8} + \frac{2x^3}{3} + x^2 - \frac{x^6}{12} \right]_0^2 = \rho \left(2 + \frac{16}{3} + 4 - \frac{16}{3} \right) = 6\rho, \\ M_{xy} &= \int_0^2 \int_{x^2}^{x+2} \int_0^x z \rho dz dy dx = \rho \int_0^2 \int_{x^2}^{x+2} \frac{x^2}{2} dy dx = \rho \int_0^2 \frac{x^2}{2}(x + 2 - x^2) dx \\ &= \rho \left[\frac{x^4}{8} + \frac{x^3}{3} - \frac{x^5}{10} \right]_0^2 = \rho \left(2 + \frac{8}{3} - \frac{32}{10} \right) = \frac{22}{15}\rho. \end{aligned}$$

Therefore,

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{44\rho}{15} \times \frac{3}{8\rho}, 6 \times \frac{3}{8\rho}, \frac{22\rho}{15} \times \frac{3}{8\rho} \right) = \left(\frac{11}{10}, \frac{9}{4}, \frac{11}{20} \right).$$



Online 4

A mass is occupying a plane region D . Let I denote the moment of inertia about the origin and I_m be the moment of inertia about the center of mass of D . Show that

$$I = I_m + d^2 M,$$

where d is the distance of the center of mass to the origin and M is the mass.

sol.

Let $\rho(x, y)$ be the density function, and (\bar{x}, \bar{y}) be the center of mass. So d can be written as $d = \sqrt{\bar{x}^2 + \bar{y}^2}$. Also, from p. 1007, we know that

$$\begin{aligned} I &= \int \int_D (x^2 + y^2) \rho(x, y) dA \\ I_m &= \int \int_D ((x - \bar{x})^2 + (y - \bar{y})^2) \rho(x, y) dA. \end{aligned}$$

From this we have,

$$\begin{aligned} I &= \int \int_D ((x - \bar{x} + \bar{x})^2 + (y - \bar{y} + \bar{y})^2) \rho(x, y) dA \\ &= \int \int_D ((x - \bar{x})^2 + (y - \bar{y})^2 + 2\bar{x}(x - \bar{x}) + 2\bar{y}(y - \bar{y}) + \bar{x}^2 + \bar{y}^2) \rho(x, y) dA \\ &= I_m + 2\bar{x} \int \int_D x \rho(x, y) dA - 2\bar{x}^2 \int \int_D \rho(x, y) dA \\ &\quad + 2\bar{y} \int \int_D y \rho(x, y) dA - 2\bar{y}^2 \int \int_D \rho(x, y) dA + (\bar{x}^2 + \bar{y}^2) \int \int_D \rho(x, y) dA. \end{aligned}$$

Note that

$$\begin{aligned} M &= \int \int_D \rho(x, y) dA, \\ \int \int_D x \rho(x, y) dA &= \bar{x} M \quad \text{and} \quad \int \int_D y \rho(x, y) dA = \bar{y} M. \end{aligned}$$

Thus,

$$\begin{aligned} I &= I_m + 2\bar{x}^2 M - 2\bar{x}^2 M + 2\bar{y}^2 M - 2\bar{y}^2 M + d^2 M \\ &= I_m + d^2 M \end{aligned}$$

as stated.

Note:

- Some students use the change of variables:

$$\tilde{x} = x + \bar{x} \quad \text{and} \quad \tilde{y} = y + \bar{y},$$

which is also a good way to solve this problem. However, be careful that in this way, the density function should be written in the new coordinate too.

Online 5

Let E be the solid in the first octant bounded by $y + z = 2$ and $2x = y$. Evaluate

$$\int \int \int_E xe^z dV$$

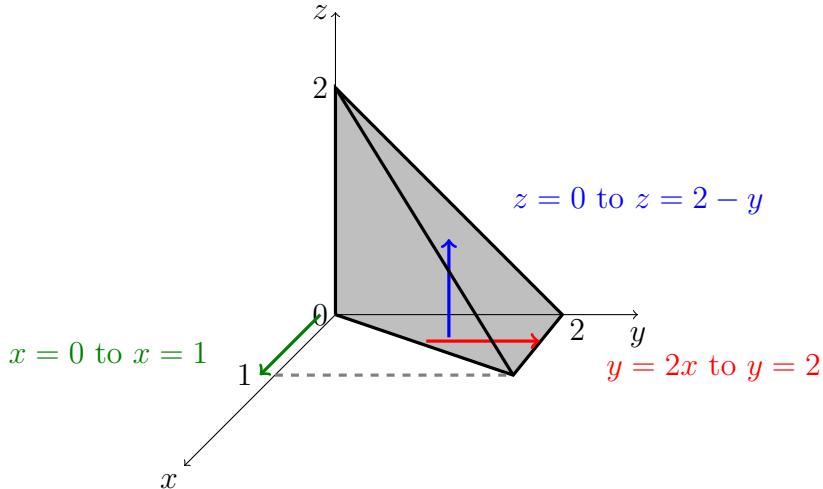
by integrating first over the projection of E onto the (a) xy -plane; (b) xz -plane; (c) yz -plane.
sol.

(a)

$$\begin{aligned} \int_0^1 \int_{2x}^2 \int_0^{2-y} xe^z dz dy dx &= \int_0^1 \int_{2x}^2 (xe^{2-y} - x) dy dx \\ &= \int_0^1 x \left[-e^{2-y} - y \right]_{2x}^2 dx = \int_0^1 x(-3 + 2x + e^{2-2x}) dx \end{aligned}$$

For the third term, use integration by parts with $u = x$ and $dv = e^{2-2x}$. This yields

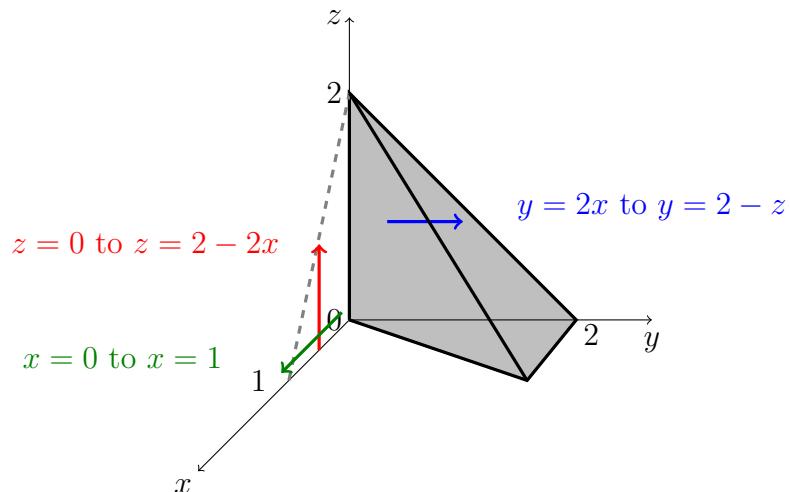
$$\begin{aligned} &\int_0^1 x(-3 + 2x) dx + \int_0^1 xe^{2-2x} dx \\ &= \left[-\frac{3x^2}{2} + \frac{2x^3}{3} + x \cdot \frac{-1}{2}e^{2-2x} - 1 \cdot \frac{1}{4}e^{2-2x} \right]_0^1 \\ &= -\frac{5}{6} - \frac{1}{2} - \frac{1}{4} + \frac{e^2}{4} = \frac{-19}{12} + \frac{e^2}{4}. \end{aligned}$$



(b)

$$\begin{aligned}
& \int_0^1 \int_0^{2-2x} \int_{2x}^{2-z} xe^z dy dz dx \\
&= \int_0^1 \int_0^{2-2x} xe^z (2 - z - 2x) dz dx \\
&= \int_0^1 \int_0^{2-2x} ((2x - 2x^2)e^z - xze^z) dz dx \\
&= \int_0^1 \left[(2x - 2x^2)e^z - xze^z + xe^z \right]_0^{2-2x} \\
&= \int_0^1 ((2x - 2x^2)e^{2-2x} - x(2 - 2x)e^{2-2x} + xe^{2-2x} - 2x + 2x^2 - x) dx \\
&= \int_0^1 (xe^{2-2x} - 3x + 2x^2) dx = \frac{-19}{12} + \frac{e^2}{4}.
\end{aligned}$$

The last equation is because we obtained exactly the same formula that already computed in (a).



(c)

$$\begin{aligned} \int_0^2 \int_0^{2-y} \int_0^{\frac{y}{2}} xe^z dx dz dy &= \int_0^2 \int_0^{2-y} \left[\frac{x^2}{2} e^z \right]_0^{\frac{y}{2}} dz dy \\ &= \int_0^2 \int_0^{2-y} \frac{y^2}{8} e^z dz dy = \int_0^2 \frac{y^2}{8} (e^{2-y} - 1) dy \end{aligned}$$

For the first term, use integration by parts with $u = \frac{y^2}{8}$ and $dv = e^{2-y}$. This yields

$$\begin{aligned} &\int_0^2 \frac{y^2}{8} (e^{2-y} - 1) dy \\ &= \left[\frac{y^2}{8} (-e^{2-y}) - \frac{y}{4} e^{2-y} + \frac{1}{4} (-e^{2-y}) - \frac{y^3}{24} \right]_0^2 \\ &= -\frac{1}{2} - \frac{1}{2} - \frac{1}{4} - \frac{1}{3} + \frac{e^2}{4} = \frac{-19}{12} + \frac{e^2}{4}. \end{aligned}$$

