Calculus(II) 0412 Homework 10 Part I

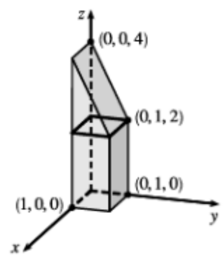
__Textbook 15.1.13__

Evaluate the double integral by first identifying it is the volume of a solid:

$$\int\int_R (4-2y)dA, \qquad R=[0,1]\times[0,1]$$

sol.

This integral represents the volume of the trapezoidal cylinder:



where the rectangular solid $[0,1]\times[0,1]\times[0,4]$ is cut by the plane $z=4-2y$.
Hence, the integral is

$$\int\int_R (4-2y)dA = \frac{2+4}{2}\cdot 1 = 3.$$

Note:

- This exercise asks to find a double integral via the volume of a solid it represented. So you shouldn't integrate it directly nor use the midpoint rule. A similar example is shown in Example 2, p. 977.

__Textbook 15.2.29__

Find the volume of the solid enclosed by the surface

$$z = x\sec^2 y$$

and the planes

$$z=0, y=0, x=2, y=0 \text{ and } y=\frac{\pi}{4}.$$

sol.

The volume is equal to the double integral:

$$\int_0^2\int_0^{\frac{\pi}{4}} x\sec^2 y dy dx$$
$$=\int_0^2 x dx\int_0^{\frac{\pi}{4}}\sec^2 y dy = \left[\frac{x^2}{2}\right]_0^2\left[\tan y\right]_0^{\frac{\pi}{4}} = 2.$$

Textbook 15.2.40

(a) In what way are the theorems of Fubini and Clairaut similar?

(b) If $f(x,y)$ is continuous on $[a,b]\times[c,d]$ and

$$g(x,y)=\int_a^x\int_c^y f(s,t)dtds$$

for $a<x<b, c<y<d$, show that $g_{xy}=g_{yx}=f(x,y)$.

sol.

(a) Fubini's theorem shows that the order of *integration* does not affect the result of the double integral, while the Clairaut's theorem shows that the order of *differentiation* does not affect the result of the second derivative.

(b) From the Fundamental Theorem of Calculus (FTC), we have

$$g_x=\frac{\partial}{\partial x}\left[\int_a^x\int_c^y f(s,t)dtds\right]=\int_c^y f(x,t)dt.$$

Applying the FTC again on y gives

$$g_{xy}=\frac{\partial}{\partial y}\left[\int_c^y f(x,t)dt\right]=f(x,y).$$

According to the *Fubini's theorem*, the order of integration in $g(x,y)$ can be exchanged:

$$g(x,y)=\int_a^x\int_c^y f(s,t)dtds=\int_c^y\int_a^x f(s,t)dsdt.$$

Similarly, applying the FTC on the above equation yields

$$g_y=\frac{\partial}{\partial y}\left[\int_c^y\int_a^x f(s,t)dsdt\right]=\int_a^x f(s,y)ds$$

and therefore,

$$g_{xy}=\frac{\partial}{\partial x}\left[\int_a^x f(s,y)ds\right]=f(x,y).$$

Hence, we conclude that $g_{xy}=g_{yx}=f(x,y)$.

Note:

- Some students might get confuse with the word "similar" in part (a). In fact, this means "looks alike" but not "the same". So the point is just to state that this two theorems both show that the order of operations is exchangeable. Although they both need continuity, this is not the reason why we want to compare them since many other theorems also require continuity!

- Be careful of the notation $\frac{d}{dx}$ and $\frac{\partial}{\partial x}$. Only when the function g is of single variable can we use the notation $\frac{dg}{dx}$ or g'. Otherwise we should use the partial derivative $\frac{\partial g}{\partial x}$ or g_x.

Online 1

Evaluate the double integral

$$\int\int_{[1,3]\times[0,1]} ye^{xy} dA$$

sol.

Clearly, it is easier to deal with x first:

$$\begin{aligned}\int_0^1 \left(\int_1^3 ye^{xy}dx\right)dy &= \int_0^1 y\left[\frac{1}{y}e^{xy}\right]_1^3 dy\\ &= \int_0^1 (e^{3y}-e^y)dy = \left[\frac{1}{3}e^{3y}-e^y\right]_0^1\\ &= \frac{1}{3}e^3 - e - \frac{1}{3} + 1 = \frac{1}{3}e^3 - e + \frac{2}{3}\end{aligned}$$

Note:

- Some students have misunderstood the boundary of the integral. You would not lose any point if you have done that $\int_1^3\int_0^1 ye^{xy}dxdy = e^3 - e - 2$. However, please remember that $[a,b]\times[c,d]$ usually represents $[x_1,x_2]\times[y_1,y_2]$ but not $[y_1,y_2]\times[x_1,x_2]$.

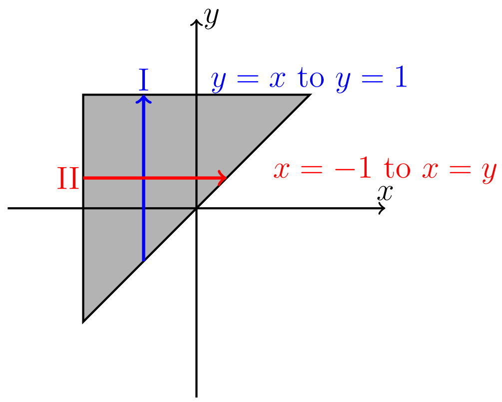
Online 2

Consider

$$\int\int_D (xy - y^3)dA,$$

where D is the region enclosed by $y = x, x = -1$ and $y = 1$. Evaluate the integral twice, first by considering D as plane region of type I and then by considering D as plane region of type II.

sol.



Type I:

$$\begin{aligned}
&\int_{-1}^{1}\left(\int_{x}^{1}(xy-y^3)dy\right)dx\\
&=\int_{-1}^{1}\left[\frac{x}{2}y^2-\frac{y^4}{4}\right]_{x}^{1}dx\\
&=\int_{-1}^{1}\left(\frac{x}{2}-\frac{1}{4}-\frac{x^3}{2}+\frac{x^4}{4}\right)dx\\
&=\left[\frac{-x}{4}+\frac{x^2}{4}-\frac{x^4}{8}+\frac{x^5}{20}\right]_{-1}^{1}\\
&=-\frac{2}{5}.
\end{aligned}$$

Type II:

$$\begin{aligned}
&\int_{-1}^{1}\left(\int_{-1}^{y}(xy-y^3)dx\right)dy\\
&=\int_{-1}^{1}\left[\frac{yx^2}{2}-xy^3\right]_{-1}^{y}dy\\
&=\int_{-1}^{1}\left(\frac{y^3}{2}-y^4-\frac{y}{2}-y^3\right)dy\\
&=\left[-\frac{y^4}{8}-\frac{y^5}{5}-\frac{y^2}{4}\right]_{-1}^{1}\\
&=-\frac{2}{5}.
\end{aligned}$$

Note:

- We only ask for the "integral" but not the "volume". So you don't need to consider the absolute value of it.

- Be careful about the direction of integration. We usually integrate **from the left to the right** and **from the bottom to the top**. Otherwise you might get an answer with the opposite sign. For example, if you put the boundary of x upside down in II, you would find that

$$\int_{-1}^{1}\left(\int_{y}^{-1}(xy-y^3)dx\right)dy=\frac{2}{5}.$$