Convergence of Sums of Dependent Bernoulli Random Variables: An Application from Portfolio Theory

Madelyn Houser,

Dept. of Mathematical Sciences,

University of Delaware,

mhouser@udel.edu

Pak-Wing Fok[*],

Ewing Hall 412,

Dept. of Mathematical Sciences,

University of Delaware,

pakwing@udel.edu

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Abstract

Generalizations of the Central Limit Theorem to N dependent random variables often assume that the dependence falls off as $N \to \infty$. In this paper we present an example from mathematical finance where convergence is *independent* of N. Specifically, we consider $N \gg 1$ dependent Bernoulli variates that represent N loans and

[*]Corresponding Author

probabilistic dependence among loans is based on an underlying economic model for default. A simple model for correlated default is the Vasicek Asymptotic Single Risk Factor (ASRF) framework. Our results showcase an example of “fast” convergence of dependent variates to this limiting non-normal distribution with rate $O(N^{-1})$.

Keywords: Central Limit Theorem; Dependent Random Variables; Credit Portfolio; Vasicek;

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1 Introduction

A fundamental result in probability theory is the Central Limit Theorem (CLT) (Gnedenko et al., 1954; Petrov, 2012; Feller, 1971) which finds numerous applications in statistical physics, engineering and operations research. DeMoivre put forth the first version of the CLT, proving it for the case of independent Bernouilli random variables. Laplace generalized the theorem to binomial distributions in 1812, and then to general distributions; however, a rigorous proof of a general Central Limit Theorem was not obtained until the 20th century by Alexander Liapounoff. One form of the theorem can be stated as:

Let $X_1,\ X_2,\cdots,X_N$ be a sequence of independent and identically distributed random variables, each having mean $\mu_1 < \infty$ and variance $\sigma^2 < \infty$. Then the distribution of $W_N \equiv \frac{X_1+\ldots+X_N-N\mu_1}{\sqrt{N}\sigma}$ tends to a unit normal distribution as $N \to \infty$. Specifically, if $F_{W_N}(x)$ is the CDF of W_N, then

$$\lim_{N\to\infty} \max_{-\infty<x<\infty} |F_{W_N}(x) - \Phi(x)| = 0 \tag{1}$$

where $\Phi(x)$ is the CDF of a standard normal distribution.

Many other mathematicians have also dedicated their energies to studying the convergence rate of W_N: what is the error in approximating W_N with a normal random variable? Berry (1941) and Esseen (1942) independently calculated the bound for the error in the CDF, thereby quantifying the "rate" at which W_N tends to normality. The Berry-Esseen Theorem states that for $-\infty < x < \infty$,

$$|F_{W_N}(x) - \Phi(x)| \leq \frac{k\mu_3/\mu_2}{\sigma\sqrt{N}}, \tag{2}$$

for some constant k, and μ_2, μ_3 are the absolute second and third moments of X_k, assumed to be non-zero. The $\sqrt{N}$ in the denominator of (2) implies that the convergence of F_{W_N} to a normal CDF occurs with rate $O(N^{-1/2})$.

The basic version of the CLT assumes that variates are independent and identically distributed (iid) with finite second moment. Researchers have also found CLT-type results when variates are not iid. For example, Lindeberg proved a more general version of the CLT for non-identically distributed, independent random variables (Le Cam, 1986). However, less is known about the convergence of variates that are not independent. Here, the main goal is to find the conditions under which summands converge to the attractor (Tikhomirov, 1981) and characterizing the dependence structure of the variates becomes crucial. Conditions such as strong mixing (Rosenblatt, 1956) and m-dependence (Stein et al., 1972) give mathematical rigor to the notion that summands must become "less dependent" as N gets large.

In this paper we present a simple case of N *dependent* Bernoulli random variables where we can easily calculate the limiting (non-normal) distribution. The convergence of such random variables finds applications in credit portolio theory where Bernoulli models for correlated default are common. The dependence structure is independent of N and stems from an underlying economic model. We confirm that the limiting distribution is the one derived by Vasicek (1987) in the Asymptotic Single Risk Factor (ASRF) framework. The CDF for the ASRF – and attractor for the dependent Bernoulli variates – is a composition of an error function with a scaled and shifted inverse error function. Using results from Esseen and Laplace's method, we show that the rate of convergence to the ASRF is $O(N^{-1})$.

2 A Family of Dependent Bernoulli Variables

2.1 Background on Credit Portfolios

A credit portfolio is a group of loans or lines of credit issued by a bank or financial institution (Bluhm et al., 2016). Such portfolios can generate interest, but this aspect of the portfolio is ignored in the following analysis which focuses on the *loss* incurred due to borrower default. A default occurs when the assets of a borrowing company in the portfolio fall below a threshold level and they are unable to pay back the loan.

When a borrower defaults on the loan, the lender incurs a loss. In some cases, the lender may recuperate a percentage of the loan, so the loss is regarded as a fraction of the amount lent. For N loans, define R_i as the percentage loss given default (LGD) of the ith borrower, $i = 1, \ldots, N$. We will assume that the LGD is 100% of the loan amount: the lender loses the entire amount when the borrower defaults (the extension to other LGDs is straightforward). Therefore we model the R_i as Bernoulli random variables and define the *loss* of the portfolio over a given holding period by

$$R_N^* = \sum_{i=1}^{N} w_i R_i, \tag{3}$$

where w_i are called *exposures* that represent the amount lent to borrower i, as a fraction of the total dollar amount invested in the portfolio: the w_i are positive weights that sum to one.

The problem that is of interest to mathematical financiers is to calculate the limiting distribution of (3) given some probabilistic economic model about how borrowers default. These limiting distributions facilitate the calculation of financial indicators such as Expected Shortfall and Value-at-Risk, important metrics for assessing the capital needed to cover losses during periods of extreme financial stress (Goodhart, 2011; Linsmeier and Pearson, 2000).

For simplicity, we will assume uniform exposures $w_i = 1/N$ so that according to eq. (3), the total loss is simply the fraction of companies that default and

$$R_N^* = \frac{1}{N} \sum_{i=1}^{N} R_i(z_i), \tag{4}$$

where R_i are identical Bernoulli random variables which describe whether a company has defaulted or not ($R_i = 1 \iff i$th borrower defaults), according to their *asset returns* z_i.

The asset returns of borrowing companies are linked to each other and the greater economy in a complex system of correlations, meaning companies in a portfolio may be more likely to default in unison. This correlation is described through a structural model in which companies are exposed to shared *risk factors*. The structural model can give rise to complicated dependencies among borrowers: see for example (Fok et al., 2014; Schonbucher, 2001; Pykhtin, 2004). In the simplest model, each company's assets are represented as the sum of an idiosyncratic risk factor and a global risk factor. The global risk factor represents stochasticity in the general economy, affecting all borrowers since "a rising tide lifts all boats." Thus, the asset returns z_i, $i = 1, 2, \ldots, N$, are modeled as (Vasicek, 1987)

$$z_i = \sqrt{\rho}\hat{\epsilon} + \sqrt{1-\rho}\epsilon_i, \qquad 0 < \rho < 1, \tag{5}$$

where $\hat{\epsilon} \sim \mathcal{N}(0,1)$, $\epsilon_i \sim \mathcal{N}(0,1)$, and ρ is the correlation of assets between any two companies and is assumed to be positive. Eq. (5) can also be thought of an implicit copula model based on the multivariate Gaussian distribution; such applications of copulas to credit portfolio modeling are widely used in risk management (De Servigny and Renault, 2004).

2.2 Dependence Structure

We now introduce a threshold θ_i so that $z_i < \theta_i \iff i$th borrower defaults, where $z_i \in \mathbb{R}$ follows (5). Actual values of θ_i would depend on the credit rating of the borrower; borrowers with better credit scores are less likely to default on their loan and would have more negative θ_i. Again for simplicity, we will assume $\theta_i = \theta$ so all borrowers are equally credit-worthy. Then $z_i \big| \hat{\epsilon} \sim \mathcal{N}(\sqrt{\rho}\hat{\epsilon}, 1-\rho)$ and

$$\begin{aligned} p(\hat{\epsilon}) &\equiv P(z_i < \theta \big| \hat{\epsilon}), & (6)\\ &= \int_{-\infty}^{\theta} \frac{1}{\sqrt{2\pi(1-\rho)}} \exp\left\{-\frac{(t-\sqrt{\rho}\hat{\epsilon})^2}{2(1-\rho)}\right\} dt, & (7)\\ &= \Phi\left(\frac{\theta - \sqrt{\rho}\hat{\epsilon}}{\sqrt{1-\rho}}\right). & (8)\end{aligned}$$

Conditioned on the global risk $\hat{\epsilon}$, $\{R_1, R_2, \ldots, R_N\}$ are iid Bernoulli($p(\hat{\epsilon})$) random variables:

$$R_i(z_i \big| \hat{\epsilon}) = \begin{cases} 0, & \text{with probability } 1 - p(\hat{\epsilon}), \\ 1, & \text{with probability } p(\hat{\epsilon}). \end{cases} \tag{9}$$

Before proceeding further, it is convenient to introduce notations relating to the standard normal distribution:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy. \tag{10}$$

The unconditional default probability of any borrower is $\int_{-\infty}^{\infty} p(x)\phi(x)dx = \Phi(\theta)$, so

$$R_i = \begin{cases} 0, & \text{with probability } 1 - \Phi(\theta), \\ 1, & \text{with probability } \Phi(\theta), \end{cases} \tag{11}$$

and the covariance of borrowers i and j is

$$\begin{aligned} \mathrm{Cov}(R_i, R_j) &= \int_{-\infty}^{\infty} p^2(x)\phi(x)dx - \Phi(\theta)^2, & (12)\\ &= \Phi(\theta)(1 - \Phi(\theta)) - 2T\left(\theta, \sqrt{\frac{1-\rho}{1+\rho}}\right), & (13)\end{aligned}$$

where $T(h,a) = \frac{1}{2\pi}\int_0^a \frac{\exp\left[-\frac{1}{2}h^2(1+x^2)\right]}{1+x^2}dx$ is Owen's T-function (Owen, 1956). When $\rho = 1$, $z_i = \hat{\epsilon}$, the Bernoulli variates are copies of each other and $\mathrm{Cov}(R_i, R_j) = \Phi(\theta)(1-\Phi(\theta))$ as expected. When $\rho = 0$, $z_i = \epsilon_i$, the Bernoulli variates are independent, Owen's function has the property $T(\theta, 1) = \frac{1}{2}\Phi(\theta)(1-\Phi(\theta))$, and $\mathrm{Cov}(R_i, R_j) = 0$.

2.3 Loss Distribution for Large N

From (4), $R_N^*|\hat{\epsilon}$ is an empirical average of N independent Bernoulli variates. By the Strong Law of Large Numbers,

$$R_N^*|\hat{\epsilon} \;=\; p(\hat{\epsilon}), \qquad N \to \infty, \tag{14}$$

almost surely, and $R_N^*|\hat{\epsilon}$ is a random variable whose outcome depends on the draw of $\hat{\epsilon}$. So

$$\begin{aligned} P[R_\infty^* \le x] &= P\left[p(\hat{\epsilon}) \le x\right], & (15)\\ &= P\left[\hat{\epsilon} \ge \frac{\theta - \sqrt{1-\rho}\,\Phi^{-1}(x)}{\sqrt{\rho}}\right] & (16)\\ \Rightarrow F_{R_\infty^*}(x) &= \Phi\left[\frac{\sqrt{1-\rho}\,\Phi^{-1}(x) - \theta}{\sqrt{\rho}}\right]. & (17) \end{aligned}$$

This limiting distribution for the mean of N dependent Bernoulli variates following (11) with covariance structure (13) is the closed-form solution to the Vasicek Asymptotic Single Risk Factor (ASRF) model. Eq. (17) provides an analytic approximation of market risks associated with a credit portfolio of N loans and is the starting point for calculations of Value-at-Risk, Expected Shortfall and other financial metrics.

3 Convergence to the Limiting Distribution

We now show that the convergence of R_N^* to its CDF (17) occurs with rate $O(N^{-1})$. Our starting point is the theorem proved by (Esseen, 1945) which gives the explicit form of the correction term when applying the Central Limit Theorem to $R_N^*|\hat{\epsilon}$:

Theorem 3.1. *Let $X_1,\ X_2,\cdots,X_N$ be a sequence of independent and identical random variables with mean zero, variance $\sigma^2 \neq 0$, and third moment μ_3. Suppose further that X_i is a lattice random variable so that its CDF has discontinuities separated by a distance d. Let F_{W_N} be the CDF of $W_N \equiv \frac{\sum_{i=1}^N X_i}{\sigma\sqrt{N}}$, which has zero mean and unit variance. Then*

$$F_{W_N}(x) = \Phi(x) + \frac{\mu_3(1-x^2)e^{-x^2/2}}{6\sigma^3\sqrt{2\pi N}} + \frac{d \cdot \psi_N(x)e^{-x^2/2}}{\sigma\sqrt{2\pi N}} + o(N^{-1/2}), \tag{18}$$

where

$$\psi_N(x) = Q\left(\frac{(x-\xi_N)\sigma\sqrt{N}}{d}\right), \qquad Q(x) = [x] - x + \frac{1}{2}. \tag{19}$$

The parameter ξ_N is the location of the smallest non-negative discontinuity of $F_{W_N}(x)$ and $[\cdot]$ is the floor function.

We now apply this theorem to find the distribution of R_N^*. We first center the Bernoulli variates so they have zero mean: $X_i = R_i - p$. Then

$$W_N = \frac{\sum_{i=1}^N X_i}{\sigma\sqrt{N}} = \frac{\sqrt{N}R_N^*}{\sigma} - \frac{\sqrt{N}p}{\sigma}, \tag{20}$$

The variable W_N has support on $\left\{\frac{-\sqrt{N}p}{\sigma} + \frac{k}{\sigma\sqrt{N}}\right\}$, for $k \in \{0, 1, \cdots, N\}$. The location of the smallest non-negative discontinuity corresponds to $k = \lceil\kappa\rceil$ where κ satisfies

$$\frac{-\sqrt{N}p}{\sigma} + \frac{\kappa}{\sigma\sqrt{N}} = 0 \Longrightarrow \kappa = Np, \tag{21}$$

and so $\xi_N = \frac{-Np+[Np]+1}{\sigma\sqrt{N}}$. Because R_N^* and W_N are related through (20) and $\sigma^2 = p(1-p)$,

$d = 1/(\sigma\sqrt{N})$,

$$\begin{aligned} F_{R_N^*|\hat\epsilon}(x) = \quad & \Phi\left(\frac{(x-p)\sqrt{N}}{\sqrt{p(1-p)}}\right) + \frac{(p-3p^2+2p^3)}{6\sqrt{N}(p(1-p))^{3/2}}\left(1-\frac{(x-p)^2N}{p(1-p)}\right)e^{-\frac{(x-p)^2N}{2p(1-p)}} \\ & + \frac{1}{\sqrt{2\pi Np(1-p)}}Q\left(xN-1-[Np]\right)e^{-\frac{(x-p)^2N}{2p(1-p)}} + o(N^{-1/2}). \end{aligned} \tag{22}$$

To find the unconditioned CDF, we multiply by $\phi(\hat\epsilon)$ and integrate over all values of $\hat\epsilon$:

$$\begin{aligned} \int_{-\infty}^{\infty} F_{R_N^*|\hat\epsilon}(x)\phi(\hat\epsilon)d\hat\epsilon \quad = \quad & \int_{-\infty}^{\infty}\Phi\left(\frac{(x-p)\sqrt{N}}{\sqrt{p(1-p)}}\right)\phi(\hat\epsilon)d\hat\epsilon + I_N(x) \\ & + (\text{Higher Order Terms in } N)\,. \end{aligned} \tag{23}$$

where

$$I_N(x) = I_N^{(1)}(x) + I_N^{(2)}(x) + I_N^{(3)}(x), \tag{24}$$

with

$$\begin{aligned} I_N^{(1)}(x) &= \frac{1}{\sqrt{N}}\int_{-\infty}^{\infty}\alpha_1(\hat\epsilon)e^{-N\tau(x,\hat\epsilon)}d\epsilon, & \alpha_1(\hat\epsilon) &= \frac{\left[p(\hat\epsilon)-3p(\hat\epsilon)^2+2p(\hat\epsilon)^3\right]\phi(\hat\epsilon)}{6\left[p(\hat\epsilon)(1-p(\hat\epsilon))\right]^{3/2}}, \\ I_N^{(2)}(x) &= \sqrt{N}\int_{-\infty}^{\infty}\alpha_2(x,\hat\epsilon)e^{-N\tau(x,\hat\epsilon)}d\epsilon, & \alpha_2(x,\hat\epsilon) &= \frac{\left[p(\hat\epsilon)-3p(\hat\epsilon)^2+2p(\hat\epsilon)^3\right]\tau(x,\hat\epsilon)\phi(\hat\epsilon)}{3(p(\hat\epsilon)(1-p(\hat\epsilon)))^{3/2}}, \\ I_N^{(3)}(x) &= \frac{1}{\sqrt{N}}\int_{-\infty}^{\infty}\alpha_3(x,\hat\epsilon)e^{-N\tau(x,\hat\epsilon)}d\epsilon, & \alpha_3(x,\hat\epsilon) &= \frac{Q\left(Nx-1-[Np(\hat\epsilon)]\right)\phi(\hat\epsilon)}{\sqrt{2\pi p(\hat\epsilon)(1-p(\hat\epsilon))}}, \end{aligned} \tag{25}$$

and

$$\tau(x,\hat\epsilon) = \frac{(x-p(\hat\epsilon))^2}{2p(\hat\epsilon)(1-p(\hat\epsilon))}. \tag{26}$$

3.1 Error incurred by Laplace's Method

First we consider the $\int_{-\infty}^{\infty}\Phi\phi d\hat\epsilon$ term in (23). The lemma below shows that it is asymptotically equivalent to Vasicek's CDF, eq. (17), with an error that scales as $\mathcal{O}(N^{-1})$. We informally call this error the "Laplace error" to distinguish it from the error incurred by applying the Central Limit Theorem in section 3.2

Lemma 3.2. *For* $N \gg 1$, $\int_{-\infty}^{\infty} \Phi\left(\frac{(x-p)\sqrt{N}}{\sqrt{p(1-p)}}\right)\phi(\hat{\epsilon})d\hat{\epsilon}$ *is asymptotically equal to Vasicek's CDF as given by eq. (17):*

$$\int_{-\infty}^{\infty} \Phi\left(\frac{(x-p)\sqrt{N}}{\sqrt{p(1-p)}}\right)\phi(\hat{\epsilon})d\hat{\epsilon} = \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(x)-\theta}{\sqrt{\rho}}\right) + E_N(x), \tag{27}$$

as $N \to \infty$, *where* $E_N(x) = \mathcal{O}(N^{-1})$.

Proof. Let $V_N(x) = \int_{-\infty}^{\infty} \Phi\left(\frac{(x-p)\sqrt{N}}{\sqrt{p(1-p)}}\right)\phi(\hat{\epsilon})d\hat{\epsilon}$. Then using Laplace's method,

$$\begin{aligned} V_N'(x) &= \sqrt{N}\int_{-\infty}^{\infty} e^{-\frac{N(x-p)^2}{2p(1-p)}} \frac{\phi(\hat{\epsilon})}{\sqrt{2\pi p(1-p)}}d\hat{\epsilon}. && (28)\\ &= \alpha(\hat{\epsilon}^*)\exp\{-N\tau(\hat{\epsilon}^*)\}\sqrt{\frac{2\pi}{\tau''(\hat{\epsilon}^*)}} + \mathcal{O}(N^{-1}), \qquad N\to\infty && (29)\end{aligned}$$

where

$$\tau(\hat{\epsilon}) = \frac{(x-p(\hat{\epsilon}))^2}{2p(\hat{\epsilon})(1-p(\hat{\epsilon}))}, \qquad \alpha(\hat{\epsilon}) = \frac{\phi(\hat{\epsilon})}{\sqrt{2\pi p(\hat{\epsilon})(1-p(\hat{\epsilon}))}}. \tag{30}$$

Note that for all $\hat{\epsilon}$, $\tau(\hat{\epsilon}) \geq 0$ so the minimum of $\tau(\hat{\epsilon})$ occurs when $x = p(\hat{\epsilon})$; thus the minimum of $\tau(\hat{\epsilon})$ is zero and $\hat{\epsilon}^*$ satisfies $p(\hat{\epsilon}^*) = x \Rightarrow \hat{\epsilon}^* = \frac{\sqrt{1-\rho}\Phi^{-1}(x)-\theta}{-\sqrt{\rho}}$. With $\tau(\hat{\epsilon}^*) = 0$, $\tau''(\hat{\epsilon}^*) = \frac{p'(\hat{\epsilon}^*)^2}{p(\hat{\epsilon}^*)(1-p(\hat{\epsilon}^*))} > 0$,

$$\begin{aligned} V_N'(x) &= \sqrt{\frac{1-\rho}{\rho}}\exp\left\{\frac{1}{2}\left[\Phi^{-1}(x)\right]^2 - \frac{\left[\sqrt{1-\rho}\Phi^{-1}(x)-\theta\right]^2}{2\rho}\right\} + \mathcal{O}(N^{-1}),\\ \Rightarrow V_N(x) &= \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(x)-\theta}{\sqrt{\rho}}\right) + E_N(x).\end{aligned}$$

where

$$
\begin{aligned}
E_N(x) &= \frac{1}{N}\int_0^x \sqrt{\frac{2\pi}{\tau''(\hat{\epsilon}^*)}}\left\{\frac{\alpha''(\hat{\epsilon}^*)}{2\tau''(\hat{\epsilon}^*)} + \frac{\alpha(\hat{\epsilon}^*)\tau^{(4)}(\hat{\epsilon}^*)}{8[\tau''(\hat{\epsilon}^*)]^2} + \frac{\alpha'(\hat{\epsilon}^*)\tau'''(\hat{\epsilon}^*)}{2[\tau''(\hat{\epsilon}^*)]^2} - \frac{5[\tau'''(\hat{\epsilon}^*)]^2\alpha(\hat{\epsilon}^*)}{24[\tau''(\hat{\epsilon}^*)]^3}\right\}dx' + o(N^{-1}) \\
&= \mathcal{O}(N^{-1}).
\end{aligned}
$$

The expression for E_N comes from higher order corrections to Laplace's method (Bender and Orszag, 1999). □

3.2 Error incurred by Central Limit Approximation

Now consider the $I_N(x)$ term in eq. (23). We will use Laplace's method to show that the scalings for $I_N^{(k)}$ are all $\mathcal{O}(N^{-1})$. As before, let $\hat{\epsilon}^*$ be the minimizer of $\tau(\hat{\epsilon})$ so that $\tau(\hat{\epsilon}^*) = 0$, $\tau'(\hat{\epsilon}^*) = 0$, $\tau''(\hat{\epsilon}^*) > 0$. For $N \to \infty$, we approximate $I_N^{(1)}$ and $I_N^{(2)}$ using Laplace's method:

$$
\begin{aligned}
I_N^{(1)}(x) &\sim \frac{\alpha_1(\hat{\epsilon}^*)}{N}\sqrt{\frac{2\pi}{\tau''(\hat{\epsilon}^*)}}, \\
I_N^{(2)}(x) &\sim \frac{\sqrt{2\pi}}{N}\left[-\frac{\alpha_2''(\hat{\epsilon}^*)}{2\tau''(\hat{\epsilon}^*)^{3/2}} + \frac{\alpha_2'(\hat{\epsilon}^*)\tau'''(\hat{\epsilon}^*)}{2\tau(\hat{\epsilon}^*)^{5/2}}\right].
\end{aligned}
$$

Since $\alpha_2(\hat{\epsilon}^*) = 0$, we have used the higher-order terms from (Bender and Orszag, 1999) for the Laplace approximation of $I_N^{(2)}$. The treatment of $I_N^{(3)}$ differs slightly because $Q\left(Nx - 1 - [Np(\hat{\epsilon})]\right)$ has discontinuities. Our solution is to first bound $I_N^{(3)}$ using Cauchy-Schwarz before applying

Laplace's method:

$$
\begin{aligned}
\left|I_N^{(3)}(x)\right| &\leq \int_{-\infty}^{\infty}\left|\frac{Q\left(Nx-1-[Np(\hat{\epsilon})]\right)}{\sqrt{2\pi Np(\hat{\epsilon})(1-p(\hat{\epsilon}))}}e^{-N\tau(\hat{\epsilon})}\phi(\hat{\epsilon})\right|d\hat{\epsilon} \\
&\leq \int_{-\infty}^{\infty}\left|Q\left(Nx-1-[Np(\hat{\epsilon})]\right)\right|\left|\frac{\phi(\hat{\epsilon})}{\sqrt{2\pi Np(\hat{\epsilon})(1-p(\hat{\epsilon}))}}\right|e^{-N\tau(\hat{\epsilon})}d\hat{\epsilon} \\
&\leq \frac{1}{2}\int_{-\infty}^{\infty}\frac{\phi(\hat{\epsilon})}{\sqrt{2\pi Np(\hat{\epsilon})(1-p(\hat{\epsilon}))}}e^{-N\tau(\hat{\epsilon})}d\hat{\epsilon} \\
&\sim \frac{1}{2}\frac{\phi(\hat{\epsilon}^*)}{\sqrt{2\pi Np(\hat{\epsilon}^*)(1-p(\hat{\epsilon}^*))}}\sqrt{\frac{2\pi}{N\tau''(\hat{\epsilon}^*)}}\exp\{-N\tau(\hat{\epsilon}^*)\} \\
&= \frac{\phi(\hat{\epsilon}^*)}{2N\left|p'(\hat{\epsilon}^*)\right|},
\end{aligned}
$$

as $N \to \infty$ and we used $\tau''(\hat{\epsilon}^*) = \frac{p'(\hat{\epsilon}^*)^2}{p(\hat{\epsilon}^*)(1-p(\hat{\epsilon}^*))}$. Therefore $I_N = I_N^{(1)}+I_N^{(2)}+I_N^{(3)} = \mathcal{O}(N^{-1})$. This prediction is confirmed by our numerical experiments in Fig. **??**, where we plot $||I_N(x)||_\infty$ against the true error in the Vasicek CDF (17). While these errors are similar, they are not identical (even for an infinite number of Monte Carlo trials) because Vasicek's CDF only approximates the $\int_{-\infty}^{\infty}\Phi\phi d\hat{\epsilon}$ term in (23).

4 Conclusions

Limiting distributions for sums of random variables remain a vital part of probability theory. They are also important in the area of mathematical finance, a source of interesting examples where the random variables are dependent and the dependence structure is motivated by an economic model. In this paper, we found the limiting distribution, eq. (17), of the mean of N dependent Bernoulli variates as $N \to \infty$. We found that the total error of the CDF (17) is comprised of two terms: the error associated with using Laplace's method to produce the Vasicek approximation (section 3.1) and the error associated with approximating the CDF using the central limit theorem (section 3.2). Both errors scale as $\mathcal{O}(N^{-1})$. Therefore, the total error of the Vasicek approximation is $\mathcal{O}(N^{-1})$ and can be summarized as

$$\begin{aligned}\underbrace{\int_{-\infty}^{\infty} F_{R_N^*|\hat{\epsilon}}(x)\phi(\hat{\epsilon})d\hat{\epsilon}}_{\text{Exact CDF}} &\sim \int_{-\infty}^{\infty} \Phi\left(\frac{(x-p)\sqrt{N}}{\sqrt{p(1-p)}}\right)\phi(\hat{\epsilon})d\hat{\epsilon} + \underbrace{I_N(x)}_{\text{Central Limit error}} \\ &\sim \underbrace{\Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(x)-\theta}{\sqrt{\rho}}\right)}_{\text{Vasicek approximation}} + \underbrace{E_N(x)}_{\text{Laplace error}} + \underbrace{I_N(x)}_{\text{Central Limit error}},\end{aligned} \tag{31}$$

as $N \to \infty$. Of course, when the Bernoulli variates are independent, the convergence to normality is $\mathcal{O}(N^{-1/2})$ by the Berry-Esseen inequality (2). In this paper, we presented a specific example of dependent variates where the convergence is "faster", at $O(N^{-1})$. We verified this scaling both analytically and numerically.

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