DISTRIBUTIONS DERIVED FROM THE NORMAL DISTRIBUTION

Definition: A random variable $X$ with pdf
\[ g(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \quad x \geq 0 \]
has **gamma distribution** with parameters $\alpha > 0$ and $\lambda > 0$.

The gamma function $\Gamma(x)$, is defined as
\[ \Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du. \]

Properties of the Gamma Function:
(i) $\Gamma(x + 1) = x\Gamma(x)$
(ii) $\Gamma(n + 1) = n!$
(iii) $\Gamma(1/2) = \sqrt{\pi}$.

Remarks:
1. Notice that an exponential rv with parameter $1/\theta = \lambda$ is a special case of a gamma rv. with parameters $\alpha = 1$ and $\lambda$.
2. The sum of $n$ independent identically distributed (iid) exponential rv, with parameter $\lambda$ has a gamma distribution, with parameters $n$ and $\lambda$.
3. The sum of $n$ iid gamma rv with parameters $\alpha$ and $\lambda$ has gamma distribution with parameters $n\alpha$ and $\lambda$.

Definition: If $Z$ is a standard normal rv, the distribution of $U = Z^2$ called the **chi-square distribution with 1 degree of freedom**.
The density function of $U \sim \chi^2_1$ is
\[ f_U(x) = \frac{x^{-1/2}}{\sqrt{2\pi}} e^{-x/2}, \quad x > 0. \]

Remark: A $\chi^2_1$ random variable has the same density as a random variable with gamma distribution, with parameters $\alpha = 1/2$ and $\lambda = 1/2$.

Definition: If $U_1, U_2, \ldots, U_k$ are independent chi-square rv-s with 1 degree of freedom, the distribution of $V = U_1 + U_2 + \ldots + U_k$ is called the **chi-square distribution with $k$ degrees of freedom**.

Using Remark 3. and the above remark, a $\chi^2_k$ rv. follows gamma distribution with parameters $\alpha = k/2$ and $\lambda = 1/2$. Thus the density function of $V \sim \chi^2_k$ is:
\[ f_V(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2}, \quad x > 0. \]
PROPOSITION: If $V$ has a chi-square distribution with $k$ degree of freedom, then

$$E(V) = k, \quad \text{Var}(V) = 2k.$$ 

DEFINITION: If $Z \sim \mathcal{N}(0, 1)$ and $U \sim \chi^2_n$ and $Z$ and $U$ are independent, then the distribution of

$$Z \sqrt{U/n}$$

is called the $t$ distribution with $n$ degrees of freedom.

PROPOSITION: The density function of the $t$ distribution with $n$ degrees of freedom is

$$f(t) = \frac{\Gamma[(n + 1)/2]}{\sqrt{n\pi} \Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}.$$ 

REMARKS: For the above density $f(t) = f(-t)$, so the $t$ density is symmetric about zero. As the number of degrees of freedom approaches infinity, the $t$ distribution tends to be standard normal.

DEFINITION: Let $U$ and $V$ be independent chi-square variables with $m$ and $n$ degrees of freedom, respectively. The distribution of

$$W = \frac{U/m}{V/n}$$

is called $F$ distribution with $m$ and $n$ degrees of freedom and is denoted by $F_{m,n}$.

REMARKS:

(i) If $T \sim t_n$, then $T^2 \sim F_{1,n}$.

(ii) If $X \sim F_{n,m}$, then $X^{-1} \sim F_{m,n}$. 

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