1 Enumeration

A set is a collection of elements. Two sets have the same cardinality if and only if they can be put in one-to-one correspondence.

Example 1 Let $A = \{ \text{cat, dog, bird} \}$ and $B = \{ \text{sing, dance, trumpet} \}$. Sets $A$ and $B$ have the same cardinality. The mapping below witnesses the one-to-one correspondence.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>cat</td>
<td>dance</td>
</tr>
<tr>
<td>dog</td>
<td>sing</td>
</tr>
<tr>
<td>bird</td>
<td>trumpet</td>
</tr>
</tbody>
</table>

Example 2 Let $C$ be the students enrolled in this class. There is no one-to-one correspondence between set $C$ and set $A$. Therefore, the two sets do not have the same cardinality.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>cat</td>
<td>David</td>
</tr>
<tr>
<td>dog</td>
<td>Adam</td>
</tr>
<tr>
<td>bird</td>
<td>Leanne</td>
</tr>
<tr>
<td>Xin</td>
<td>...</td>
</tr>
</tbody>
</table>

Note we do not have to count the elements in either set. All we have to do is check to see whether any mapping exists which puts the elements of one set into one-to-one correspondence with the other. For those familiar with properties of functions, this means the mapping must be one-to-one and onto (i.e. it is both injective and surjective; i.e. it is a bijection). For this reason, we can apply the notion of “same cardinality” to infinite sets.

The natural numbers are 0, 1, 2, 3, .... We use the symbol $\mathbb{N}$ to indicate the natural numbers.

Example 3 Let $\text{EVEN}$ be the set of even numbers. The set $\text{EVEN}$ and $\mathbb{N}$ have the same cardinality as witnessed by the following mapping.

<table>
<thead>
<tr>
<th>$\mathbb{N}$</th>
<th>$\text{EVEN}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

On the one hand, this is surprising. But it is clear that we have a function which maps distinct elements $\mathbb{N}$ to distinct elements in $\text{EVEN}$. Also, this function maps every element in $\mathbb{N}$ to one in $\text{EVEN}$. Also, this function makes sure that for every element in $\text{EVEN}$, there is some number in $\mathbb{N}$ which gets mapped to it. Since this function exists, the two sets have the same cardinality.
Sets that have the same cardinality as \( \mathbb{N} \) are called countably infinite and enumerable.

**Example 4** The rational numbers are those which can be expressed as fractions. Let \( \mathbb{Z} \) denote this set. \( \mathbb{Z} \) and \( \mathbb{N} \) have the same cardinality. Drawing this on a Cartesian plane helps one see the idea. In fact, this shows that the set of all pairs of numbers are in one-to-one correspondence with \( \mathbb{N} \).

**Exercise 1** Let \( \Sigma \) denote a finite alphabet and \( \Sigma^* \) the set of all strings of finite length constructable from this alphabet. Think about \( \Sigma^* \) for a minute. Is this set countably infinite?

Are there any infinite sets that are not countably infinite? Surprisingly, the answer is yes. This drove Cantor mad, though nowadays we’re used to the idea. Remember, the irrationality of the square root of two drove Pythagoras and his followers crazy too.

The power set of a set is the set of all subsets of it. Recall set \( A = \{ \text{dog, bird, cat} \} \). The powerset of \( A \) contains all the subsets of \( A \). We write

\[
P(A) = \begin{cases} \emptyset & \{\text{dog, cat}\} \\
\{\text{dog}\} & \{\text{dog, bird}\} \\
\{\text{cat}\} & \{\text{bird, cat}\} \\
\{\text{bird}\} & \{\text{dog, bird, cat}\} 
\end{cases}
\]

We can now identify an infinite set that is not countably infinite.

**Theorem 1** (Cantor) \( P(\mathbb{N}) \) is not countably infinite.

**Proof.** For contradiction, suppose that it is. Then there is some mapping which puts \( \mathbb{N} \) into one-to-one correspondence with \( P(\mathbb{N}) \). Call this mapping \( M \). So \( M \) maps 0 to some subset of \( \mathbb{N} \), and 1 to a different subset, and so on. Now consider the following subset of \( \mathbb{N} \):

\[
S = \{ n \in \mathbb{N} \mid n \notin M(n) \}
\]

Question: Which \( n \) maps to \( S \)? Consider any \( n \in \mathbb{N} \). If it belongs to \( M(n) \) then it is not in \( S \). Therefore \( S \neq M(n) \). If it does not belong to \( M(n) \) then it is in \( S \). Again \( S \neq M(n) \). In either case, they are not the same. Since we considered any such \( n \), it follows that no \( n \in \mathbb{N} \) maps \( n \) to the set \( S \). \( \square \)

Since the powerset of \( \mathbb{N} \) is infinite, but not countably infinite, it must be of a higher order of infinitude. It can be shown that the real numbers have the same cardinality as \( P(\mathbb{N}) \). And thus there are strictly more real numbers than natural numbers (though both are infinite). Such sets are said to be uncountably infinite.