1. (a) (2 pts) The ceiling of $x$, denoted $\lceil x \rceil$, is the unique integer $n$ such that $n \leq x < n + 1$. True  False (Circle the correct answer).

(b) (2 pts) The external path length of a binary tree with $N$ internal nodes is the internal path length plus $2N$. True  False

(c) (2 pts) The usual recursive C++ function to traverse a binary tree in preorder is tail recursive. True  False.

(d) (2 pts) For any algorithm $A$ and a decomposition of the inputs $\mathcal{I} = \bigcup_{n=1}^{\infty} S_n$, $t_{\bar{A}}(n) \leq t_{A}(n)$. True  False.

(e) (4 pts) Give a simple formula for $1 + 2 + 3 + \cdots + n$.

(f) (4 pts) What is the external path length of the binary tree shown below?

(g) (2 pts) A binary tree with $N$ internal nodes has exactly $N - 1$ external nodes. True  False

(h) (2 pts) If an array $A[1], \ldots, A[N]$ contains a heap, the parent of $A[k]$ is found in $A[k/2]$. True  False

(i) (4 pts) Give a simple formula for $1 + x + x^2 + \cdots + x^n$ for $x \neq 1$. 
2. (5 pts) Draw a full binary tree of height 3 with a minimum number of nodes that is not a complete tree.

3. A three-tree is an ordered tree consisting of two types of nodes: external nodes with no children and internal nodes with exactly three children. A full three-tree is one in which the internal nodes completely fill every level, except possibly the last.

(a) (4 pts) Draw a complete three-tree of height 2.

(b) (4 pts) What is the maximum number of nodes that can occur at level \( \ell \) in a three-tree?

(c) (6 pts) What are the maximum and minimum number of nodes in a full three-tree of height \( h \geq 0 \)?

(d) (6 pts) Derive a formula (in terms of \( N \)) for the height of a full three-tree containing \( N \) nodes.
4. Below are the two possible heaps containing the values 1, 2, 3.

(a) (2 pts) How many possible heaps are there containing 1, 2, 3, 4?
(b) (4 pts) Draw them.

(c) (4 pts) How many possible heaps are there containing 1, 2, 3, 4, 5?

5. Draw a binary search tree containing 1, 2, 3, 4, 5

(a) (3pts) of minimum height

(b) (3pts) of maximum height.

6. (5 pts) Give a definition for the abstract data type dictionary.
7. (10 pts) Write a C++ function `node* search( keyType v, node *L )` that searches a list \( L \) and returns \( t \) such that \( v == t->key \). Your code should use a sentinel to speed up the search. Assume that the list \( L \) has a head node, an end node \( z \), and the following node structure:

```cpp
typedef int keyType; // Type for the key in a list node

struct node {
    keyType key;
    node* next;

    // Constructor design especially for initializing z.
    node(): next(this){}

    node(keyType v, node *p): key(v), next(p){}
};

node *z = new node; // Global z node
```
8. (20 pts)

(a) Write a recursive C++ function int height(node *t) that computes the height of the binary tree pointed to by t. Assume that t points to a header node and external nodes are represented by the z node. An empty tree has height minus one. Assume the following definition for the struct node.

```cpp
struct node          // node structure for binary tree
{                   
    char info;      
    node *l, *r;   

    node()          // Constructor needed to initialize const z. 
    { l = r = this; } 
};                 // end declaration of struct node
```

```cpp
node* z = new node;  // Declaration of global z.
```

(b) What is the maximum computing time of your function? Use the “big oh” notation in your answer; be sure to specify the meaning of all symbols used.