Aided Inertial Navigation: Unified Feature Representations and Observability Analysis

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References
Abstract

Extending our recent work [1] that focuses on the observability analysis of aided inertial navigation systems (INS) using homogeneous geometric features including points, lines and planes, in this paper, we complete the analysis for the general aided INS using different combinations of geometric features (i.e., points, lines and planes). We analytically show that the linearized aided INS with different feature combinations generally possess the same observability properties as those with same features, i.e., 4 unobservable directions, corresponding to the global yaw rotation and the global position of the sensor platform. During the analysis, we particularly propose a novel minimal representation of line features, i.e., the “closest point” parameterization, which uses a 4D Euclidean vector to describe a line and is proved to preserve the same observability properties. Based on that, for the first time, we provide two sets of unified representations for points, lines and planes, i.e., the quaternion form and the closest point (CP) form, and perform extensive observability analysis with analytically-computed Jacobians for these unified parameterizations. We validate the proposed CP representations and observability analysis with Monte-Carlo simulations, in which EKF-based vision-aided INS (VINS) with combinations of geometrical features in CP form are developed and compared.

1 Introduction and Related Work

Inertial navigation systems (INS) have become ubiquitous and found their ways in many application domains from augmented reality to autonomous navigation. However, low-cost inertial measurement units (IMUs) suffer from non-negligible measurement noises and time-varying biases. Pose estimation solely based on IMU measurements (angular velocity and linear acceleration) will result in large errors in a short period of time. For this reason, great efforts have been devoted to INS aided with additional sensors such as camera [2] and sonar [3], in order to achieve higher localization accuracy.

As low-cost and light-weight optical cameras are one of the ideal aiding sources for INS, many different state estimation algorithms for vision-aided INS (VINS) have been developed (e.g., [4, 5, 6, 7, 8, 9, 10] and references therein). Among them, extended Kalman filter (EKF)-based methods are among the most popular, such as multi-state constrained Kalman filter (MSCKF) [2], observability constrained (OC)-EKF [4, 11], right invariant error (RI)-EKF [12].

System observability plays important roles for state estimation [13]. First of all, understanding a system observability provides a deep insight about the system’s geometrical properties [14, 4, 15] and determines the minimal measurement modalities or state parameters needed to initialize an estimator. Secondly, it can be used to identify degenerate motions [1, 16] that cause additional unobservable directions and should be avoided or alerted in real-world applications. Thirdly, the observability-based methodologies used in OC-EKF [11] and OC-VINS [4] that enforce the correct observability properties, can be adopted to improve estimation consistency. Last but not least important, analytical measurement Jacobians for aided INS estimators, can be verified through the observability analysis process.

For these reasons, in the past decade, significant research efforts have been devoted on the observability analysis for INS. Among them, Martinelli [17] has proved that biases, velocity, and roll and pitch angles in VINS are observable, while Hesch et al. [4] and Kottas et al. [18] have analytically derived the observability matrix for linearized VINS and shown that there are 4 unobservable directions for VINS with point features. However, the vast majority of VINS estimators and their related observability analysis have focused only on point features, albeit using line [19, 20, 21, 22, 23] and plane [24, 25, 26, 27] features has recently been reviving. Different parameterizations of line features in VINS have been considered. A unit quaternion with a distance scalar to represent line features was used in VINS [20] where observability analysis for linearized VINS with lines was
also performed. Plücker representation with orthonormal error states for lines [28] were adopted in point and line based VINS [21, 22]. Two ending points of line segment to represent line features were used in recent point/line-based MSCKF [23]. On the other hand, plane features were also exploited in recent SLAM and VINS systems. For example, the VINS observability with point features and plane features with known orientation was studied in [25], and the 2D LiDAR aided INS (L-INS) was also developed by estimating the perpendicular structural planes within structured buildings [24]. The dense planar visual-inertial SLAM system has recently developed [26] which incorporates plane features extracted from dense point clouds and represented in quaternion form as in [29]. In contrast, in our prior work [30], we have introduced the closest point (CP) representation of planes for aided INS, which was successfully employed in our recent 3D LiDAR-inertial plane SLAM system [27].

One particular challenge when incorporating different geometrical features into state estimation is to find appropriate representations, especially for their error states. As seen from the above discussion, while different parameterizations for point, line and planes have been used, there is still a lack of an unified representation for all these geometrical features, which is of particular interest when using all different features in a single estimator. To this end, Nardi et al. [31] have tried to find a unified representation for different geometrical primitives with a uniform data structure and the operation between different geometrical features can be decided by a multiplication table. However, it is not clear how to incorporate measurement uncertainty into this representation, which is necessary for state estimation.

In this paper, to bridge this gap, by fully understanding the most commonly used parameterizations for different geometrical features, we propose two sets of unified representations for point, line and plane: (i) the unit quaternion based parameterization, and (ii) the closest point based parameterization. Note that each unified representation has a minimal feature error state and the related measurement uncertainties can be easily incorporated into an INS estimator. Moreover, based on our recent work [1] where we performed the observability analysis for linearized aided INS with homogeneous point, line and plane features separately, we analyze the observability for linearized aided INS with combinations of point, line and plane features all in our proposed CP form and examine their corresponding unobservable subspace. Specifically, the main contributions of this paper include:

- We introduce two sets of unified representations of both quaternion and CP forms for points, lines and planes. In particular, the CP form of lines is proposed for the first time, which is a minimal parameterization with a 4D Euclidean vector to represent a line and its error state.
- We perform complete observability analysis for aided INS with points, lines and planes in quaternion (or CP) form and show that the aided INS have the same unobservable properties given the analytically computed measurement Jacobians with the proposed unified parameterization. We also prove that aided INS with combination of all the geometrical features in CP form generally have the same 4 unobservable directions.
- We conduct extensive Monte-Carlo simulations to validate the proposed CP representation and observability analysis for the aided INS with combinations of geometrical features.

2 Aided Inertial Navigation System

In this section, we briefly describe the system and measurement models for the aided INS with different geometrical features. The state vector of the aided INS contains the current IMU state...
\( x_I \) and the feature state \( Gx_{\text{feat}} \):

\[
x = \begin{bmatrix} x_I^T & Gx_{\text{feat}}^T \end{bmatrix}^T = \begin{bmatrix} [I_G \bar{q}]^T & b_g^T & Gv_I^T & b_a^T & Gp_I^T & Gx_{\text{feat}}^T \end{bmatrix}^T
\]

where \( I_G \bar{q} \) is a unit quaternion represents the rotation between the current IMU frame \( \{I\} \) and the global frame \( \{G\} \), and \( I_G \mathbf{R}(\bar{q}) \) is the corresponding rotation matrix. \( b_g \) and \( b_a \) represent the gyroscope and accelerometer biases, respectively, while \( Gv_I \) and \( Gp_I \) denote the current IMU velocity and position in the global frame. \( Gx_{\text{feat}} \) generically denotes the features of different geometric types such as points, lines and planes. For simplicity, the aiding exteroceptive sensor frame is assumed to coincide with the IMU frame in this paper.

### 2.1 System Kinematic Model

The motion model of the system can be described as [32]:

\[
\begin{aligned}
\dot{I_G \bar{q}}(t) &= \frac{1}{2} \Omega(I(G) \bar{q}(t)), \quad Gp_I(t) = Gv_I(t), \quad G\dot{v}_I(t) = Ga(t) \\
b_g(t) &= n_{wg}(t), \quad b_a(t) = n_{wa}(t), \quad G\dot{x}_{\text{feat}}(t) = 0_{m \times 1}
\end{aligned}
\]

where \( \omega \) and \( a \) are the angular velocity and linear acceleration, respectively. \( n_{wg} \) and \( n_{wa} \) are the zero-mean Gaussian noises driving the IMU gyroscope and accelerometer biases. \( m \) is the dimension of \( Gx_{\text{feat}} \), and \( \Omega(\omega) \equiv \begin{bmatrix} -[\omega]^T & \omega & 0 \end{bmatrix} \), with \([·]\) denotes skew symmetric matrix. Hence, the linearized continuous-time error-state equations can be written as:

\[
\dot{\bar{x}}(t) \approx \begin{bmatrix} F_c(t) & 0_{15 \times m} \end{bmatrix} \bar{x}(t) + \begin{bmatrix} G(t) \\ 0_{m \times 12} \end{bmatrix} n(t)
\]

\[
= F(t)\bar{x}(t) + G(t)n(t)
\]

where \( \bar{x} = x - \hat{x} \) denotes the errors between the true state \( x \) and its estimate \( \hat{x} \). The error states for quaternion can be described as: \( \delta \bar{q} = \bar{q} \otimes \bar{q}^{-1} \) with \( \otimes \) represents the quaternion multiplication and \( \delta \bar{q} = \begin{bmatrix} 1 & \delta \theta^T \\ 0 \end{bmatrix}^T \). \( F_c(t) \) and \( G_c(t) \) are the continuous-time error-state transition matrix and noise Jacobian matrix, respectively. \( n(t) = [n_g^T \, n_{wg} \, n_a]^T \) is modeled as a zero-mean Gaussian process with autocorrelation \( E\left[n(t)n^T(\tau)\right] = Q_c \delta(t - \tau) \). Note that \( n_g(t) \) and \( n_a(t) \) represent the Gaussian process with autocorrelation \( \delta(t - \tau) \).

The discrete-time state transition matrix \( \Phi_{(k+1,k)} \) from time \( t_k \) to \( t_{k+1} \) can be derived as:

\[
\Phi_{(k+1,k)} = \begin{bmatrix} \Phi_I & 0_{15 \times m} \\ 0_{m \times 15} & \Phi_{\text{feat}} \end{bmatrix}
\]

\[
= \begin{bmatrix} \Phi_{I11} & \Phi_{I12} & 0_3 & 0_3 & 0_3 & 0_{3 \times m} \\ 0_3 & I_3 & 0_3 & 0_3 & 0_3 & 0_{3 \times m} \\ \Phi_{I31} & \Phi_{I32} & I_3 & \Phi_{I34} & 0_3 & 0_{3 \times m} \\ 0_3 & 0_3 & 0_3 & I_3 & 0_3 & 0_{3 \times m} \\ \Phi_{I51} & \Phi_{I52} & \Phi_{I53} & \Phi_{I54} & I_3 & 0_{3 \times m} \\ 0_{m \times 3} & 0_{m \times 3} & 0_{m \times 3} & I_{m \times 3} & 0_{m \times 3} & I_m \end{bmatrix}
\]

where \( \Phi_I \) and \( \Phi_{\text{feat}} \) represents the state transition matrix of the IMU state and the feature state respectively, and the analytical solution can be found in [4].
2.2 Point Measurements

The point measurements from different sensors in the aided INS can be generically modeled as range and/or bearing measurements [1]:

\[
\mathbf{z}_p = \begin{bmatrix} z^{(r)} \\ z^{(b)} \end{bmatrix} = \begin{bmatrix} \sqrt{\mathbf{I}_{xf}^\top \mathbf{I}_{xf} + n^{(r)}} \\ h_b(\mathbf{I}_{xf}, n^{(b)}) \end{bmatrix} \approx \begin{bmatrix} \mathbf{H}_r^\top \mathbf{\dot{x}}_f + n^{(r)} \\ \mathbf{H}_b^\top \mathbf{\dot{x}}_f + \mathbf{H}_n n^{(b)} \end{bmatrix} \tag{5}
\]

where \( \mathbf{I}_{xf} \) represents a 3D point in IMU frame, \( \mathbf{H}_r \) and \( \mathbf{H}_b \) are the range and bearing measurement Jacobians with respect to \( \mathbf{I}_{xf} \), \( \mathbf{H}_n \) is the noise Jacobian, \( n^{(r)} \) and \( n^{(b)} \) are zero-mean Gaussian noises for range and bearing measurements, respectively. To keep presentation concise, we also define \( \mathbf{H}_{proj} = [\mathbf{H}_r^\top \mathbf{H}_b^\top]^\top \) (see [1]).

2.3 Line Measurements

Without loss of generality, we consider the projective line measurements [28], which can be defined as the distance of two ending image points, \( \mathbf{x}_s \) and \( \mathbf{x}_e \), to the projected line segment in the image:

\[
\mathbf{z}_l = \begin{bmatrix} \frac{\mathbf{x}_s^\top l}{\sqrt{l_1^2 + l_2^2}} \\ \frac{\mathbf{x}_e^\top l}{\sqrt{l_1^2 + l_2^2}} \end{bmatrix}^\top \tag{6}
\]

where \( l = [l_1 \ l_2 \ l_3]^\top \) is the projected 2D image line from 3D line \( \mathbf{l}_{xf} \) in the IMU frame.

2.4 Plane Measurements

Given point cloud from sensors (e.g., RGBD camera or LiDAR), we can directly extract planes. Therefore, we assume a direct plane measurement model as:

\[
\mathbf{z}_\pi = \mathbf{l}_{xf} + n^{(\pi)} \tag{7}
\]

where \( \mathbf{I}_{xf} \) is the plane in the IMU frame and \( n^{(\pi)} \) is the zero-mean white Gaussian noise.

2.5 Observability Analysis

Observability analysis for the linearized aided INS can be performed in a similar way as in [11, 4]. In particular, the observability matrix \( \mathbf{M}(\mathbf{x}) \) is given by:

\[
\mathbf{M}(\mathbf{x}) = \begin{bmatrix} \mathbf{H}_{x,1} \Phi_{(1,1)} \\ \vdots \\ \mathbf{H}_{x,k} \Phi_{(k,1)} \end{bmatrix} \tag{8}
\]

where \( \mathbf{H}_{x,k} \) represents the measurement Jacobians at time-step \( k \). The right null space of \( \mathbf{M}(\mathbf{x}) \), denoted by \( \mathbf{N} \), indicates the unobservable directions of the underlying system. The procedures for finding \( \mathbf{N} \) can be summarized as Fig. 1.

3 Point, Line and Plane Representation

As proper representations of point, line and plane features are important for state estimation, based on the extensive review of the most commonly used point, line and plane representations as summarized in Table 1 and Fig. 2, we introduce two sets of unified representations for points, lines and planes, i.e., the quaternion and CP parameterizations.
Figure 1: Procedures for observability analysis.

State Definition $\mathbf{x}$

State Transition Matrix $\Phi$

Measurement Jacobians $\mathbf{H}_x$

Observability Matrix $\mathbf{M}$

Find $\mathbf{N}$ so that $\mathbf{MN} = 0$

Figure 2: Geometrical parameters for (a) point $\mathbf{x}_f$, (b) line $\mathbf{x}_l$, and (c) plane $\mathbf{x}_\pi$. 
### Table 1: Summary of Point, Line and Plane Representations

<table>
<thead>
<tr>
<th>Model #</th>
<th>Point</th>
<th>Error states</th>
<th>Line</th>
<th>Error states</th>
<th>Plane</th>
<th>Error states</th>
</tr>
</thead>
<tbody>
<tr>
<td>1: General Form</td>
<td>$f_1, f_2, f_3, f_4$</td>
<td>not minimal</td>
<td>$n_l, v_l$</td>
<td>$\delta \theta_l, \delta \phi_l$</td>
<td>$\pi_1, \pi_2, \pi_3, \pi_4$</td>
<td>not minimal</td>
</tr>
<tr>
<td>2: Geometrical Form</td>
<td>$b_f, r_f$</td>
<td>not minimal</td>
<td>$n_e = n_l / |n_l|, v_e = v_l / |v_l|, d_l = n_l / |v_l|$</td>
<td>not minimal</td>
<td>$n_{\pi}, d_{\pi}$</td>
<td>not minimal</td>
</tr>
<tr>
<td>3: Spherical Form</td>
<td>$\theta_f, \phi_f, r_f$</td>
<td>$\tilde{\theta}_f, \tilde{\phi}_f, \tilde{r}_f$</td>
<td>$\theta_l, \phi_l, \alpha_l, d_l$</td>
<td>$\tilde{\theta}_l, \tilde{\phi}_l, \tilde{\alpha}_l, \tilde{d}_l$</td>
<td>$\theta_{\pi}, \phi_{\pi}, d_{\pi}$</td>
<td>$\tilde{\theta}<em>{\pi}, \tilde{\phi}</em>{\pi}, \tilde{d}_{\pi}$</td>
</tr>
<tr>
<td>4: Inverse Depth</td>
<td>$\theta_f, \phi_f, \lambda_f = \frac{1}{r_f}$</td>
<td>$\tilde{\theta}_f, \tilde{\phi}_f, \tilde{\lambda}_f$</td>
<td>$\theta_l, \phi_l, \alpha_l, \lambda_l = \frac{1}{d_l}$</td>
<td>$\tilde{\theta}_l, \tilde{\phi}_l, \tilde{\alpha}_l, \tilde{\lambda}_l$</td>
<td>$\theta_{\pi}, \phi_{\pi}, \lambda_{\pi} = \frac{1}{d_{\pi}}$</td>
<td>$\tilde{\theta}<em>{\pi}, \tilde{\phi}</em>{\pi}, \tilde{\lambda}_{\pi}$</td>
</tr>
<tr>
<td>5: Quaternion</td>
<td>$\tilde{q}_f = \frac{1}{\sqrt{1 + r_f}} \begin{bmatrix} b_f \ r_f \end{bmatrix}$</td>
<td>$\delta \theta_f$</td>
<td>$d_l, \tilde{q}_l$ with $R(\tilde{q}_l) = [n_e, v_e, [n_e]v_e]$</td>
<td>$\delta \theta_l, \tilde{d}_l$</td>
<td>$\tilde{q}<em>{\pi} = \frac{1}{\sqrt{1 + d</em>{\pi}^2}} \begin{bmatrix} n_{\pi} \ d_{\pi} \end{bmatrix}$</td>
<td>$\delta \theta_{\pi}$</td>
</tr>
<tr>
<td>6: Closest Point</td>
<td>$p_f = r_f b_f$</td>
<td>$p_f = \hat{p}_f + \tilde{p}_f$</td>
<td>$p_l = d_l \tilde{q}_l$</td>
<td>$p_l = \hat{p}_l + \tilde{p}_l$</td>
<td>$p_{\pi} = d_{\pi} n_{\pi}$</td>
<td>$p_{\pi} = \hat{p}<em>{\pi} + \tilde{p}</em>{\pi}$</td>
</tr>
</tbody>
</table>
3.1 Point Representations

Model 1 in Table 1 is the homogeneous coordinates, \( f_i \), \( i \in 1 \ldots 4 \), for a point feature \( x_f \), which is the most general form. Model 2 represents the point with a unit bearing vector \( b_f \) and a range scalar \( r_f \) measuring the distance from point to the origin \( O \) (see Fig. 2 (a)). Since the 3D unit vector \( b_f \) can be represented by 2 angles \( \theta_f \) and \( \phi_f \): \( b_f = [\cos \theta_f \cos \phi_f \sin \theta_f \cos \phi_f \sin \phi_f]^{T} \), we can easily derive Model 3, which is similar to spherical coordinate. If we use the inverse of range scalar \( \lambda_f = \frac{1}{dr_f} \), we get Model 4, which essentially is the inverse depth representation \([33]\). Recently, Maley and Huang \([34]\) introduced a unit quaternion representation (Model 5) for points, which wraps \( b_f \) and \( r_f \) into a unit quaternion. Model 5 leverages the quaternion error state, which is minimal for point state estimation. For example, a point in quaternion form can be written as:

\[
\bar{q}_f = \begin{bmatrix} q_f^T \\ q_r \end{bmatrix} = \frac{1}{\sqrt{1 + r_f^2}} \begin{bmatrix} b_f^T \\ r_f \end{bmatrix} \simeq \delta \bar{q}_f \otimes \hat{q}_f = \begin{bmatrix} \frac{1}{2} \delta \theta_f \\ 1 \end{bmatrix} \otimes \hat{q}_f
\]

(9)

where \( \delta \theta_f \) is the error state for point in unit quaternion form. Let \( p_f \) be the closest point from the point to the origin, which is the most conventional parameterization for point feature (i.e., Model 6). It can be computed by multiplying the bearing vector \( b_f \) with the range scalar \( r_f \) as:

\[
p_f = r_f b_f = \hat{p}_f + \tilde{p}_f
\]

(10)

where \( \tilde{p}_f \) is the error state in closest point form.

3.2 Line Representations

Given two points \( p_{f1} \) and \( p_{f2} \) in a line \( x_l \), we can obtain its Plücker coordinate (Model 1 of lines in Table 1) as \([28, 35]\):

\[
\begin{bmatrix} n_l \\ v_l \end{bmatrix} = \begin{bmatrix} |p_{f1}|p_{f2} \\ p_{f2} - p_{f1} \end{bmatrix}
\]

(11)

where \( n_l \) represents the normal direction of the plane constructed by the two points and the origin, \( v_l \) represents the line direction. In Model 1, the distance from the origin to the line can be computed as \( d_l = \frac{|n_l|}{|v_l|} \). Bartoli et al. \([28]\) introduced the minimal orthonormal error state (\( \delta \theta_l \) and \( \delta \phi_l \), see \([1]\) for detailed explanations) of Model 1 when involving lines in structure from motion. We can also represent the line with Model 2, which contains all the geometrical elements of a line, including a unit normal direction \( n_e = \frac{n_l}{|n_l|} \), a unit line direction \( v_e = \frac{v_l}{|v_l|} \) and the distance scalar \( d_l \) (see Fig. 2 (b)). Alternatively, a line can be parameterized by 3 angles \( \theta_l, \phi_l, \alpha_l \) and a distance \( d_l \) (Model 3, see Appendix A for the transformation between Models 2 and 3). In analogy to the case of point features, we can use the inverse depth (\( \lambda_l = \frac{1}{d_l} \)) representation (Model 4) for line features. Interestingly, Kottas et al. \([20]\) used a unit quaternion \( \bar{q}_l \) and a distance scalar \( d_l \) to represent a line (Model 5), where the quaternion describes the line direction:

\[
R(\bar{q}_l) = \begin{bmatrix} n_e & v_e & |n_e|v_e \end{bmatrix}
\]

(12)

\[
\bar{q}_l \simeq \delta \bar{q}_l \otimes \hat{q}_l = \begin{bmatrix} \frac{1}{2} \delta \theta_l \\ 1 \end{bmatrix} \otimes \hat{q}_l
\]

(13)

where \( \delta \theta_l \) represents the error state of the line quaternion. The 4D minimal error states of the line include the quaternion error angle and the distance scalar error: \([\delta \theta_l^T, \delta d_l]^T \).
More importantly, if we multiply the unit quaternion \( \bar{q}_l \) with the distance scalar \( d_l \), we obtain a 4D vector, which can be considered as the “closest point” for a line in the 4D vector space (i.e., Model 6):

\[
p_l = d_l \bar{q}_l = d_l \begin{bmatrix} q^\top_l \\ q_l \end{bmatrix}^\top = \bar{p}_l + \bar{p}_l
\]

where \( \bar{p}_l \) is the 4D error state for the closest point of a line. It should be noted that this minimal CP parameterization for lines (14) appears to be proposed for the first time and is shown to have good numerical stability (see Section 7).

3.3 Plane Representations

Similar to point features, the homogeneous coordinate \( f_i, \ i \in 1 \ldots 4 \) is the most general form for planes (Model 1) [36]. The Hesse form (Model 2) uses the normal direction \( n_\pi \) and the distance scalar \( d_\pi \) to represent a plane. As \( n_\pi \) can be represented by two angles \( \theta_\pi \) and \( \phi_\pi \) (see Fig. 2 (c)): \( n_\pi = \begin{bmatrix} \cos \theta_\pi \cos \phi_\pi \\ \sin \theta_\pi \cos \phi_\pi \\ \sin \phi_\pi \end{bmatrix}^\top \), the spherical coordinate (Model 3) can be used to represent the plane with two angles \( \theta_\pi \) and \( \phi_\pi \) and the distance scalar \( d_\pi \). If using the inverse depth \( \lambda_\pi = \frac{1}{d_\pi} \), we have the inverse depth representation for planes (Model 4). Recently, Kaess [29] proposed to use a unit quaternion to represent a plane by including the unit normal direction and the distance scalar into a quaternion (Model 5):

\[
\bar{q}_\pi = \begin{bmatrix} q_\pi \\ q_\pi \end{bmatrix} = \frac{1}{\sqrt{1 + d_\pi^2}} \begin{bmatrix} n_\pi \\ d_\pi \end{bmatrix} \simeq \delta \bar{q}_\pi \otimes \bar{q}_\pi = \begin{bmatrix} \frac{\delta \theta_\pi}{1} \\ 1 \end{bmatrix} \otimes \bar{q}_\pi
\]

where \( \delta \theta_\pi \) is the minimal error state for the quaternion plane representation. In Model 6, the closest point from the plane to the origin is used to represent the plane [37, 27] which has the minimal Euclidean error state \( \bar{p}_\pi \):

\[
p_\pi = d_\pi n_\pi = \bar{p}_\pi + \bar{p}_\pi
\]

3.4 Remarks

It is clear that Models 1 and 2 parameterizations for point, line and plane features, are not minimal, which may cause numerical issues (e.g., singular information matrices) if directly using these parameters during least-squares optimization. While Models 3 and 4 are minimal representations, these models might suffer from singularities when the elevation angle \( \phi = \pm \frac{\pi}{2} \), similar to gimbal lock for Euler angles.

Interestingly, all point [34], line [20] and plane [29] features can be parameterized by the unified representation of quaternion (Model 5), which exploits the minimal error states of quaternion for state estimation for better numerical stability. However, the observability properties of quaternion representation for point and planes are missing in the literature, though those are studied in the case of line features in [20]. Therefore, we perform an extensive observability analysis with the unified quaternion representation for points, lines and planes, where we have analytically derive the measurement Jacobians (see appendices for detailed derivations), showing that the same observability properties of aided INS are preserved.

More importantly, the closest point (CP) form (i.e., Model 6) provides another unified parameterization for different geometrical features. The CP model for point is simply its 3D position in Euclidean space. While the CP representation for planes was introduced for INS in our prior work [27], in this work, we propose a novel 4D CP model for line features with the minimal 4D
error states in 4D Euclidean space. Note that when using the CP parameterization, the error propagation for different geometrical features can be easily defined in the Euclidean vector space, and thus the cost functions of intuitive geometric interpretation can be formulated. In the following we will first perform detailed observability analysis using this novel CP line.

4 CP for Point, Line and Plane

We developed the closest point representation for point, line and plane. The closest point denotes a point in Euclidean space which is closest to the origin. We use the closest point to represent the geometrical features, such as point, line and plane. The closest point for point feature is just its 3D coordinate in Euclidean space, which is also the conventional representation for a point. We have also developed the closest point for plane representation [27, 30], which uses the nearest point from the plane to origin for parameterization of a plane feature. One advantage of closest point is that the error states are directly in Euclidean spaces, with which elegant cost function and analytic measurement Jacobians can be relatively easily formulated. Both the closest point representations for point and plane are in 3D Euclidean spaces. But for line which has 4DOF geometrical constraints, we propose to use a novel 4D Euclidean "closest point" to parametrize it.

4.1 Point in Closest Point

The closest point of point is just itself \( x_f = p_f \), which is also the conventional parameterization. We describe it as:

\[
 p_f = r_f b_f = \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix}^T
\]

(17)

Hence, the state vector with CP point can be written as:

\[
 x = \begin{bmatrix} x_I^T \\ G p_f^T \end{bmatrix}^T
\]

(18)

The measurement Jacobians based on measurement model (5) w.r.t. (18) can be described as:

\[
 H_x^{(p)} = \frac{\partial z_p}{\partial x} = \begin{bmatrix} \frac{\partial z_a}{\partial x_I} \\ \frac{\partial z_a}{\partial p_f} \end{bmatrix}
\]

(19)

where we have:

\[
 \frac{\partial z_p}{\partial x_I} = \frac{\partial z_p}{\partial I} \frac{\partial I}{\partial x_I} = H_{proj} \hat{R} \left( I_{\hat{p}_f} - G I_{\hat{p}_f} \right) G \hat{R} 0_{3x3} - I_3
\]

(20)

\[
 \frac{\partial z_p}{\partial G p_f} = \frac{\partial z_p}{\partial I} \frac{\partial I}{\partial G p_f} = H_{proj} \hat{R}
\]

(21)

Based on the state transition matrix (4), for point feature, \( \Phi_{\text{feat}} = I_3 \). We can construct the \( k \)-th block of observability matrix \( M^{(p)} \) for point measurement as:

\[
 M_k^{(p)} = H_{x_k}^{(p)} \Phi^{(p)}(k, 1) = H_{proj} \hat{R} \left[ \Gamma_{p1} \quad \Gamma_{p2} \quad \Gamma_{p3} \quad \Gamma_{p4} \quad \Gamma_{p5} \quad \Gamma_{p6} \right]
\]

(22)
where we have:

\[
\Gamma_{p1} = \left[G \dot{p}_l - G \dot{\hat{p}}_l - G \dot{v}_l \delta t_k - \frac{1}{2} G \hat{g} \delta t_k^2 \right] G \hat{R}
\]

\[
\Gamma_{p2} = \left[\left(G \dot{\hat{p}}_r - G \dot{p}_r \right) \right] G \dot{\Phi} \Phi_{112} - \Phi_{152}
\]

\[
\Gamma_{p3} = -I_3 \delta t_k
\]

\[
\Gamma_{p4} = -\Phi_{154}
\]

\[
\Gamma_{p5} = -I_3
\]

\[
\Gamma_{p6} = I_3
\]

Previous work [1, 4] has shown that the unobservable directions (the null space of \(N^{(p)}\)) for point feature can be written as:

\[
N^{(p)} = \begin{bmatrix}
I \hat{R} & G \\
0_{3 \times 1} & 0_3 \\
\left\{G \dot{v}_l\right\} G & 0_3 \\
0_{3 \times 1} & 0_3 \\
\left\{G \dot{p}_r\right\} G & I_3 \\
\left\{-G \dot{p}_l\right\} G & I_3
\end{bmatrix}
\]

(29)

### 4.2 Line in Closest Point

Since line has 4DOF, we need to find a 4-dimension Euclidean vector to describe the "closest point" of line. Based on the quaternion representation of line, the novel closest point for line is defined as:

\[
p_l = d_l \hat{q}_l = d_l \left[ q_l^\top q_l \right]^\top
\]

(30)

Thus, the state vector with CP line can be written as:

\[
x = \left[ x_l^\top \ p_l^\top \right]^\top
\]

(31)

The error states of line can also be transferred into 4D Euclidean space. The relationship between CP line error states \(
\hat{p}_l
\) and the quaternion line error states \(\left[ \delta \theta_l^\top \ \hat{d}_l \right]^\top\) can be derived as:

\[
\hat{p}_l = \hat{p}_l + \hat{\tilde{p}}_l = \left( \hat{d}_l + \hat{\tilde{d}}_l \right) \delta \hat{q}_l \otimes \hat{q} = \left( \hat{d}_l + \hat{\tilde{d}}_l \right) \mathcal{R}(\hat{q}) \left[ \frac{1}{2} \delta \theta_l \right]
\]

\[
\Rightarrow \mathcal{R}(\hat{q})^T (\hat{p}_l + \hat{\tilde{p}}_l) = \left[ \frac{1}{2} (\hat{d}_l + \hat{\tilde{d}}_l) \delta \theta_l \right] \hat{d}_l + \hat{\tilde{d}}_l \]

\[
\Rightarrow \mathcal{R}(\hat{q})^T (\hat{p}_l + \hat{\tilde{p}}_l) - \left[ 0 \right] \approx \left[ \frac{1}{2} \hat{d}_l I_3 0 \right] \left[ \delta \theta_l \right] \hat{d}_l
\]

\[
\Rightarrow \left[ \delta \theta_l \right] \hat{d}_l \approx \left[ \frac{1}{2} \hat{d}_l I_3 0 \right] \left( \mathcal{R}(\hat{q})^T (\hat{p}_l + \hat{\tilde{p}}_l) - \left[ 0 \right] \hat{d}_l \right)
\]

\[
\Rightarrow \left[ \delta \theta_l \right] \hat{d}_l \approx \left[ \frac{2}{\hat{d}_l} (\hat{q}_l I_3 - \hat{q}_l) \right] \left[ \frac{2}{\hat{q}_l} (\hat{q}_l I_3 - \hat{q}_l) \right] \hat{p}_l
\]

(32)

where \(\mathcal{R}(\cdot)\) is the right quaternion multiplication matrix [32]. Then, the measurement Jacobians based on the line measurement model (6) w.r.t. (14) can be written as:

\[
H_{x}^{(l)} = \frac{\partial z_l}{\partial x} = \left[ \frac{\partial z_l}{\partial x}, \frac{\partial z_l}{\partial y}, \frac{\partial z_l}{\partial z_l} \right]
\]

(33)
The measurement Jacobians w.r.t. the IMU state $x_I$ can be written as:

$$\frac{\partial \tilde{z}_l}{\partial x_I} = \frac{\partial \tilde{z}_l}{\partial \tilde{l}} \frac{\partial \tilde{l}}{\partial \tilde{L}} \frac{\partial \tilde{L}}{\partial x_I}$$

(34)

where we have:

$$1 = \begin{bmatrix} K & 0_3 \end{bmatrix} I_L$$

(35)

$$I_L = \begin{bmatrix} I_{n_e} \bar{d}_l \end{bmatrix} = \begin{bmatrix} \bar{R}_G^T \bar{R} \hat{p}_{II} \end{bmatrix} \begin{bmatrix} G_{n_e} \bar{d}_l \end{bmatrix}$$

(36)

$$\frac{\partial \tilde{z}_l}{\partial \tilde{l}} = \frac{1}{l_n} \begin{bmatrix} u_1 - \frac{l_1 e_1}{l_n} & v_1 - \frac{1}{l_n} & 1 \end{bmatrix}$$

(37)

$$\frac{\partial \tilde{l}}{\partial \tilde{L}} = \begin{bmatrix} K & 0_3 \end{bmatrix}$$

(38)

$$\frac{\partial \tilde{L}}{\partial x_I} = \begin{bmatrix} G_{d_l} [G_{\tilde{L}} \tilde{R} \bar{G}_{n_e}] - [G_{\tilde{L}} \hat{R} \hat{p}_{II}] G_{\bar{e}_e} \end{bmatrix} \begin{bmatrix} 0_{3 \times 9} & I_G \hat{R} \hat{G}_{\bar{e}_e} \end{bmatrix}$$

(39)

where $x_s = [u_1, v_1, 1]$, $x_e = [u_2, v_2, 1]$, $e_1 = l^x x_s$, $e_2 = l^x x_e$ and $l = [l_1, l_2, l_3]^T$. $K$ is projection matrix for line [30]. Thus, the measurement Jacobians w.r.t. the CP line feature can be written as:

$$\frac{\partial \tilde{z}_l}{\partial \tilde{q}_l} = \frac{\partial \tilde{z}_l}{\partial \tilde{l}} \frac{\partial \tilde{l}}{\partial \tilde{L}} \frac{\partial \tilde{L}}{\partial \tilde{q}_l} \frac{\partial \delta \tilde{l}}{\partial \tilde{d}_l}$$

(40)

where we have:

$$\frac{\partial \tilde{L}}{\partial \tilde{q}_l} = \begin{bmatrix} \bar{R}_G \hat{R}_{e_1} \end{bmatrix} \begin{bmatrix} G_{\tilde{L}} \hat{R}_{e_1} \end{bmatrix}$$

(41)

$$\frac{\partial \tilde{L}}{\partial \delta \tilde{d}_l} = \begin{bmatrix} \bar{R}_G \hat{p}_{II} \end{bmatrix} \begin{bmatrix} G_{\tilde{L}} \hat{p}_{II} \end{bmatrix}$$

(42)

$$\frac{\partial \delta \tilde{d}_l}{\partial \tilde{d}_l} = \begin{bmatrix} 2 (\tilde{q}_l I_3 - |\tilde{q}_l|) - 2 \tilde{q}_l \tilde{q}_l^T \end{bmatrix}$$

(43)

For line, $\Phi_{\text{feat}} = I_4$. Following the observability analysis in [4], we can have the observability matrix as:

$$M_k^{(l)} = H_{xk}^{(l)} \Phi(k, 1) = \frac{\partial \tilde{z}_l}{\partial \tilde{l}} \begin{bmatrix} K_{\tilde{G}}^T \hat{R} \hat{G}_{\bar{e}_e} \end{bmatrix} \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \Gamma_{15} & \Gamma_{16} & \Gamma_{17} \end{bmatrix}$$

(44)

where $\Gamma_{i j}, i \in 1 \ldots 2, j = 1 \ldots 7$ (see Appendix B). Therefore, we can have the following conclusions regarding the observability properties for CP line parameterization in aided INS:

**Lemma 1.** For aided INS, if there is only one line feature in the state vector, and the line feature is parameterized with the proposed 4D closest point, the system will have 5 unobservable directions.
Hence, the state vector with a CP plane in the state vector can be written as:

\[ \mathbf{N}^{(l)} = \begin{bmatrix} I \hat{R}^G \mathbf{g} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} \\ 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} \\ -[G \hat{v}_I] \mathbf{g} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} \\ 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} \\ -G \hat{n}_e \mathbf{g} & G \hat{v}_e \mathbf{g} & 0_{3 \times 1} & -G \hat{q}_l & 0_{3 \times 1} \\ -\frac{G d_\pi}{2} (G \hat{q}_l I_3 - [G \hat{\dot{q}}_l]) \mathbf{g} & \frac{1}{2} (G \hat{q}_l I_3 - [G \hat{\dot{q}}_l]) \mathbf{g} & 0_{3 \times 1} & -G \hat{q}_l & 0_{3 \times 1} \end{bmatrix} \] (45)

**Proof.** See Appendix B. \qed

Let \( \mathbf{N}_i^{(l)} \), \( i \in 1 \ldots 5 \) represent the \( i \)-th column of the \( \mathbf{N}^{(l)} \). \( \mathbf{N}_1^{(l)} \) relates to the global yaw around the gravity direction, \( \mathbf{N}_{2, 4}^{(l)} \) relate to the global position of the aided INS sensor platform, and \( \mathbf{N}_5^{(l)} \) relates to the velocity along the line direction. Therefore, we verify that the CP line keeps the observability properties for line feature in aided INS.

### 4.3 Plane in Closest Point

As in our previous work [27, 30], the closest point of the plane is the point with the smallest distance from the plane to the origin. The closest point for plane can be written as:

\[ \mathbf{p}_\pi = d_\pi \mathbf{n}_\pi \] (46)

Hence, the state vector with a CP plane in the state vector can be written as:

\[ \mathbf{x} = [X_l^T \ G \mathbf{p}_\pi^T]^T \] (47)

The measurement Jacobians based on (7) w.r.t. to (47) can be written as:

\[ \mathbf{H}_{X}^{(\pi)} = \frac{\partial \mathbf{z}_\pi}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \mathbf{z}_\pi}{\partial X_l} & \frac{\partial \mathbf{z}_\pi}{\partial \mathbf{p}_\pi} \end{bmatrix} \] (48)

The measurement Jacobians w.r.t. the IMU state \( X_l \) can be written as:

\[ \frac{\partial \mathbf{z}_\pi}{\partial X_l} = \frac{\partial \mathbf{z}_\pi}{\partial \tilde{X}_l} \begin{bmatrix} \tilde{I} \hat{\mathbf{n}}_\pi \\
{\tilde{I} \hat{d}_\pi} \end{bmatrix} \] (49)

where we have:

\[ \frac{\partial \tilde{I} \hat{d}_\pi}{\partial X_l} = \begin{bmatrix} I \hat{d}_\pi I_3 & I \hat{\mathbf{n}}_\pi \end{bmatrix} \] (50)

\[ \frac{\partial \tilde{I} \hat{\mathbf{n}}_\pi}{\partial X_l} = \begin{bmatrix} \tilde{I} \hat{R}^G \hat{\mathbf{n}}_\pi & 0_3 \\
0_{1 \times 3} & -G \hat{\mathbf{n}}_\pi^T \end{bmatrix} \] (51)
Proof. Lemma 2. where \( \Gamma \) is parameterized in closest point form, the system will have at least 7 unobservable directions as 

\[
\frac{\partial \tilde{z}_N}{\partial \mathbf{p}_\pi^G} = \frac{\partial \tilde{z}_N}{\partial \mathbf{n}_\pi^G} \begin{bmatrix} 1 & \mathbf{g} \\ \mathbf{G} & 1 \end{bmatrix} \begin{bmatrix} \Gamma^{\mathbf{G}} \mathbf{H}^{\mathbf{H}} \mathbf{G} \mathbf{H}^{\mathbf{G}} \mathbf{H} \end{bmatrix} \frac{\partial \mathbf{n}_\pi^G}{\partial \mathbf{p}_\pi^G} \quad (52)
\]

where we have:

\[
\frac{\partial \Gamma^T_{\mathbf{G}}}{\partial \mathbf{p}_\pi^G} = \begin{bmatrix} \mathbf{R}^G_{\mathbf{L}} & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{3 \times 1} & 1 \end{bmatrix} \quad (53)
\]

\[
\frac{\partial \mathbf{G}_\pi^T}{\partial \mathbf{p}_\pi^G} = \begin{bmatrix} \frac{1}{\mathbf{d}_\pi} (\mathbf{I}_3 - \mathbf{G}_\pi^T \mathbf{G}_\pi^T) \\ \mathbf{G}_\pi^T \end{bmatrix} \quad (54)
\]

For CP plane, the state transition matrix \( \mathbf{F}_{\text{cp}} = \mathbf{I} \). Following the observability analysis in [4], we can construct the \( k \)-th block of the observability matrix as:

\[
\mathbf{M}_{k}^{(\pi)} = \mathbf{H}_{kk}^{(\pi)} \mathbf{F}_{\text{cp}} = \begin{bmatrix} \frac{\partial \tilde{z}_N}{\partial \mathbf{n}_\pi^G} \mathbf{0}_{1 \times 3} [\Gamma_{\pi11} \Gamma_{\pi12} \Gamma_{\pi13} \Gamma_{\pi14} \Gamma_{\pi15} \Gamma_{\pi16}] \end{bmatrix} \quad (55)
\]

where \( \Gamma_{\pi ij}, i \in 1, 2, j \in 1 \ldots 6 \) can be found in Appendix C.

**Lemma 2.** For aided INS, if there is only one plane feature in the state vector, and the plane feature is parameterized in closest point form, the system will have at least 7 unobservable directions as \( N^{(\pi)} \).

\[
N^{(\pi)} = \begin{bmatrix} \mathbf{I}^T_{\mathbf{G}} \mathbf{R}^G_{\mathbf{L}} \mathbf{g} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} \\ -[\mathbf{G} \mathbf{P}_{\mathbf{L}}]^{G} \mathbf{g} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 1} \\ -[\mathbf{G} \mathbf{P}_{\mathbf{L}}]^{G} \mathbf{g} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} \\ -[\mathbf{G} \mathbf{P}_{\mathbf{L}}]^{G} \mathbf{g} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} \\ -[\mathbf{G} \mathbf{P}_{\mathbf{L}}]^{G} \mathbf{g} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} \\ -[\mathbf{G} \mathbf{P}_{\mathbf{L}}]^{G} \mathbf{g} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} & \mathbf{G}^{\mathbf{n}}_{\pi} \end{bmatrix} \quad (56)
\]

Proof. See Appendix C.

The \( N^{(\pi)} \) relates to the global yaw around the gravity direction, \( N^{(\pi)}_{2:4} \) relate to the aided INS sensor platform, \( N^{(\pi)}_{5:6} \) relates to the velocity parallel to the plane and \( N^{(\pi)}_{7} \) relates to the rotation around the plane normal direction. It can be seen that the CP plane representation keeps the plane observability properties in the aided INS.

### 5 Observability Analysis for Aided INS with Heterogeneous Features

In this section, we study the observability properties for the aided INS with different combinations of geometrical features including points, lines and planes (all in closest point form). To keep concise presentation, we here consider one feature of each type in the state vector.

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5.1 Point and Line Measurements

Consider a point feature $G\mathbf{p}_k$ and a line feature $G\mathbf{l}_k$ in the state vector of the aided INS:

$$\mathbf{x} = [x_f^T \ G\mathbf{p}_f^T \ G\mathbf{l}_f^T]^T$$

(57)

If the sensor observing both the point and line features, we have the following measurement model [see (5) and (6)]:

$$\mathbf{z}_{pl} = \begin{bmatrix} \mathbf{z}_p \\ \mathbf{z}_l \end{bmatrix}$$

(58)

The state transition matrix for feature will become as $\Phi_{\text{feat}} = \mathbf{I}_7$. For the observability analysis, we can compute the $k$-th block of the observability matrix as:

$$\mathbf{M}^{(pl)}_k = \left[ \mathbf{H}^{projG}_{\mathbf{p}f} \hat{\mathbf{R}} \ 0 \right] \frac{\partial \mathbf{z}_l}{\partial \mathbf{u}_k} \left[ \mathbf{K}_{\mathbf{l}k} \hat{\mathbf{R}} \ 0 \right] = \begin{bmatrix} \Gamma_{p1} & \Gamma_{p2} & \Gamma_{p3} & \Gamma_{p4} & \Gamma_{p5} & \Gamma_{p6} & 0_3 & 0_{3 \times 1} \\ \Gamma_{l1} & \Gamma_{l2} & \Gamma_{l3} & \Gamma_{l4} & \Gamma_{l5} & 0_3 & 0_{3 \times 1} & \Gamma_{l6} & \Gamma_{l7} \end{bmatrix}$$

(59)

Finally, it is not difficult to find null space (unobservable directions) as:

$$\mathbf{N}_{pl} = \begin{bmatrix} \mathbf{N}_{pl1} & \mathbf{N}_{pl2:4} \end{bmatrix} = \begin{bmatrix} \frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & 0_3 \\ 0_{3 \times 1} & -\frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & 0_3 \\ \frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & 0_3 \\ -\frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & \frac{L_3}{G} \hat{\mathbf{R}} & 0_3 \\ -\frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & 0_3 \\ \frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & \frac{L_3}{G} \hat{\mathbf{R}} & 0_3 \\ -\frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & 0_3 \\ \frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & \frac{L_3}{G} \hat{\mathbf{R}} & 0_3 \\ \vdots & \vdots & \vdots \\ \frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & 0_3 \\ -\frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & \frac{L_3}{G} \hat{\mathbf{R}} & 0_3 \\ -\frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & 0_3 \\ \frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & \frac{L_3}{G} \hat{\mathbf{R}} & 0_3 \\ \vdots & \vdots & \vdots \\ \frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & 0_3 \\ -\frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & \frac{L_3}{G} \hat{\mathbf{R}} & 0_3 \end{bmatrix} = \frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g}$$

(60)

where we define $\frac{L_1}{G} \hat{\mathbf{R}} = [G\mathbf{n}_e \ G\mathbf{v}_e [G\mathbf{n}_e \mathbf{x} G\mathbf{v}_e]$. For the aided INS with combination features of point and line, there are also 4 unobservable directions; one is $\mathbf{N}_{pl1}$ which relates the rotation around the gravity direction, the other three are $\mathbf{N}_{pl2:4}$ which correspond to the global position of the sensor platform.

We can easily extend to the case with multiple points and lines. Assume that we have $p$ points and $s$ lines in the state vector for INS, there are also 4 unobservable directions for the system and they can be written as:

$$\mathbf{N}_{pl} = \begin{bmatrix} \frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & 0_3 \\ 0_{3 \times 1} & -\frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & 0_3 \\ \frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & 0_3 \\ -\frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & \frac{L_3}{G} \hat{\mathbf{R}} & 0_3 \\ -\frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & 0_3 \\ \frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & \frac{L_3}{G} \hat{\mathbf{R}} & 0_3 \\ \vdots & \vdots & \vdots \\ \frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & 0_3 \\ -\frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & \frac{L_3}{G} \hat{\mathbf{R}} & 0_3 \\ -\frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & 0_3 \\ \frac{L_1}{G} \hat{\mathbf{R}} G \mathbf{g} & \frac{L_3}{G} \hat{\mathbf{R}} & 0_3 \end{bmatrix}$$

(61)
5.2 Point and Plane Measurements

Given that we have a point and a plane in the state vector, i.e., the feature state $x_f = \begin{bmatrix} G_p^\top & G_p^\top \pi \end{bmatrix}^\top$. Then the state vector becomes:

$$x = \begin{bmatrix} x_f^\top & G_p^\top & G_p^\top \pi \end{bmatrix}^\top \quad (62)$$

In this case, the measurement model becomes [see (5) and (7)]:

$$z_{p\pi} = \begin{bmatrix} z_p & z_{\pi} \end{bmatrix} \quad (63)$$

The state transition matrix for the feature can be written as $\Phi_{feat} = I_6$. Therefore, the $k$-th block of observability matrix can be constructed as:

$$M^{(p\pi)}_k = \begin{bmatrix} I_{kG}^\top \hat{R} & 0 \\ \delta \frac{\partial z_{p\pi}}{\partial I_k \hat{n}_{p\pi}} & I_{k1 \times 3} \end{bmatrix} \begin{bmatrix} \Gamma_{p1} & \Gamma_{p2} & \Gamma_{p3} & \Gamma_{p4} & \Gamma_{p5} & \Gamma_{p6} & 0_3 \\ \Gamma_{\pi11} & \Gamma_{\pi12} & \Gamma_{\pi13} & \Gamma_{\pi14} & \Gamma_{\pi15} & 0_3 & \Gamma_{\pi16} \\ \Gamma_{\pi21} & \Gamma_{\pi22} & \Gamma_{\pi23} & \Gamma_{\pi24} & \Gamma_{\pi25} & 0_3 & \Gamma_{\pi26} \end{bmatrix} \quad (64)$$

The unobservable directions ($N_{p\pi}$ such that $M^{(p\pi)}_k N_{p\pi} = 0$) can hence be found as:

$$N_{p\pi} = \begin{bmatrix} N_{p\pi1} & N_{p\pi2:4} \end{bmatrix} = \begin{bmatrix} I_{G}^\top \hat{R} G \mathbf{g} & 0_3 \\ 0_{3 \times 1} & 0_3 \\ -[G \mathbf{v}_{f1}]^G G \mathbf{g} & 0_3 \\ 0_{3 \times 1} & 0_3 \\ -[G \mathbf{P}_{f_k} \times] G \mathbf{g} & G_{\Pi j} \hat{R} \\ -[G \mathbf{P}_{f} \times] G \mathbf{g} & G_{\Pi j} \hat{R} \\ \vdots & \vdots \\ -[G \mathbf{P}_{f_p} \times] G \mathbf{g} & G_{\Pi n} \hat{R} \\ -[G \mathbf{P}_{\pi j}] G \mathbf{g} & G_{\Pi j} \hat{n}_{\pi} e_{3 \Pi j} \end{bmatrix}^\top \quad (65)$$

where we define $G_{\Pi j} \hat{R} = \begin{bmatrix} G \mathbf{n}_{p1}^\perp & G \mathbf{n}_{p2}^\perp & G \mathbf{n}_{p} \end{bmatrix}$. Clearly, in this case, the unobservable directions of the INS consist of the rotation around the gravity direction $N_{p\pi1}$, and the global position of the sensor platform $N_{p\pi2:4}$.

Similarly, we can extend this analysis to multiple points and planes. Given $p$ points and $n$ planes in the state vector, the null space of the observability matrix, which are also the unobservable directions, can be defined as:

$$N_{p\Pi} = \begin{bmatrix} I_{G}^\top \hat{R} G \mathbf{g} & 0_3 \\ 0_{3 \times 1} & 0_3 \\ -[G \mathbf{v}_{f1}]^G G \mathbf{g} & 0_3 \\ 0_{3 \times 1} & 0_3 \\ -[G \mathbf{P}_{f_k} \times] G \mathbf{g} & G_{\Pi j} \hat{R} \\ -[G \mathbf{P}_{f} \times] G \mathbf{g} & G_{\Pi j} \hat{R} \\ \vdots & \vdots \\ -[G \mathbf{P}_{f_p} \times] G \mathbf{g} & G_{\Pi n} \hat{R} \\ -[G \mathbf{P}_{\pi j}] G \mathbf{g} & G_{\Pi j} \hat{n}_{\pi} e_{3 \Pi j} \end{bmatrix}^\top \quad (66)$$

where $j \in \{1 \ldots n\}$. 

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5.3 Line and Plane Measurements

Assume a line and a plane in the state vector, i.e., \( x_f = [G_l^\top G_p^\top]^\top \), then the state vector can be written as:

\[
x = [x_f^\top G_p^\top G_l^\top \pi^\top]^\top
\]

The measurement model is given by [see (6) and (7)]:

\[
z_{l\pi} = [z_l \ z_{\pi}]
\]

The state transition matrix for the feature state \( \Phi_{\text{feat}} = I_7 \). The \( k \)-th block of the observability matrix is calculated as:

\[
M_{k}^{(l\pi)} = \left[ \begin{array}{ccc}
\frac{\partial z_{l\pi}}{\partial t} & \frac{\partial z_{l\pi}}{\partial \tilde{\mathbf{n}}_{\pi}} & 0 \\
0 & I_{\tilde{\mathbf{n}}_{\pi}} & 0_{3\times 1} \\
\Gamma_{l11} & \Gamma_{l12} & \Gamma_{l13} \\
\Gamma_{l21} & \Gamma_{l22} & \Gamma_{l23} \\
\Gamma_{\pi11} & \Gamma_{\pi12} & \Gamma_{\pi13} \\
\Gamma_{\pi21} & \Gamma_{\pi22} & \Gamma_{\pi23} \\
\end{array} \right] \times \\
\left[ I_{G_l^\top G_R 0} 0_{3\times 1} \right] \times \\
\left[ I_{G_l^\top G_R 0} 0_{3\times 1} \right] \times \\
\left[ I_{G_l^\top G_R 0} 0_{3\times 1} \right] \times \\
\left[ I_{G_l^\top G_R 0} 0_{3\times 1} \right]
\]

(69)

With that, we have the following result:

**Lemma 3.** For the aided INS system with one line and one plane feature in the state vector, and they are in CP form,

- If the line is parallel to the plane, the linearized system will have 5 unobservable directions as \( N_{l\pi 1:5} \).
- If the line is un-parallel to the plane, the linearized system will have 4 unobservable directions as \( N_{l\pi 1:4} \).

\[
N_{l\pi} = \left[ \begin{array}{ccc}
I_{G_l^\top G_R 0} & 0 & 0_{3\times 1} \\
0_{3\times 1} & 0 & 0_{3\times 1} \\
-\frac{G_{dl}^\top (G_q l_3 - |q_l|^2) G_{gl}}{2} & G_{l\pi} & 0_{3\times 1} \\
-\frac{G_{dl}^\top G_{ql} + G_{ql}^\top G_{dl}}{2} & G_{\tilde{\mathbf{n}}_{\pi}} & 0_{3\times 1} \\
-\frac{G_{dl}^\top G_{ql} + G_{ql}^\top G_{dl}}{2} & G_{\tilde{\mathbf{n}}_{\pi}} & 0_{3\times 1} \\
\end{array} \right]
\]

(70)

**Proof.** See Appendix E.

We can extend the above analysis to the cases with multiple lines and planes. Given that we have \( s \) lines and \( n \) planes for the estimation, unobservable directions for INS with unparalleled lines...
and plane are:

\[
N_{LII} = \begin{bmatrix}
\frac{I^G}{G}R^Gg & 0_3 \\
0_{3\times1} & 0_3 \\
-\left[G^G_V I_3 \times \right]Gg & 0_3 \\
0_{3\times1} & 0_3 \\
-\left[G^G_P I_k \times \right]Gg & 0_3 \\
0_{3\times1} & 0_3 \\
\frac{-G_{dG}}{2} \left(G^G_q I_3 - G^G_{qI} \right)Gg & \frac{1}{2} \left[G^G_{qI} I_3 - \left[G^G_{qI} \right] \right]G^G_\nu \nu \ e_0_{3\times1} - G^G_{qI} L^G_{L_k} R \\
\frac{G_{dG} I_3}{2} G^G_{qI} Gg & \frac{1}{2} \left[G^G_{qI} I_3 - \left[G^G_{qI} \right] \right] G^G_\nu \nu \ e_0_{3\times1} - G^G_{qI} L^G_{L_k} R \\
-\left[G^G_P \pi \right]Gg & -\left[G^G_P \pi \right]Gg \\
\frac{-G_{dG}}{2} \left(G^G_{qI} I_3 - G^G_{qI} \right)Gg & \frac{1}{2} \left[G^G_{qI} I_3 - \left[G^G_{qI} \right] \right] G^G_\nu \nu \ e_0_{3\times1} - G^G_{qI} L^G_{L_k} R \\
\frac{G_{dG} I_3}{2} G^G_{qI} Gg & \frac{1}{2} \left[G^G_{qI} I_3 - \left[G^G_{qI} \right] \right] G^G_\nu \nu \ e_0_{3\times1} - G^G_{qI} L^G_{L_k} R \\
\end{bmatrix}
\]

where \(i \in \{1 \ldots s\}\).

### 5.4 Point, Line and Plane Measurements

Given a point, a line and a plane in the state vector, i.e., \(x_f = [G^T p_f \ G^T p_l \ G^T p_\pi]^{\top}\). The state vector can be written as:

\[
x = [x_f^{\top} \ G^T p_f \ G^T p_l \ G^T p_\pi]^{\top}
\]

The measurement model becomes [see (5), (6) and (7)]:

\[
z_{pl\pi} = \begin{bmatrix} z_p \\ z_l \\ z_\pi \end{bmatrix}
\]

The state transition matrix for the feature state \(\Phi_{feat} = I_{10}\). Then, we can compute the \(k\)-th block of the observability matrix as follows:

\[
M^{(pl\pi)}_k = \underbrace{[H_{proj} I^G_k R \ 0 \ \partial_{z_k} I^G_k R \ 0_3]}_{\Gamma_{p1} \ \Gamma_{p2} \ \Gamma_{p3} \ \Gamma_{p4} \ \Gamma_{p5} \ \Gamma_{p6} \ \Gamma_{p7} \ \Gamma_{p8} \ \Gamma_{p9}} \times \begin{bmatrix} I^G_k R \\ 0_{3\times1} \ 1 \end{bmatrix}
\]

With that we have the following result:
Lemma 4. For the aided INS with one point, one line and one plane feature in the state vector and they are parameterized in CP form, the system will have at least 4 unobservable directions as $N_{plp}$.

\[
N_{plp} = \begin{bmatrix}
I \hat{R}^G g & 0_3 \\
0_{3 \times 1} & 0_3 \\
-\left[G \hat{V}_{l_1} \times \right]^G g & 0_3 \\
0_{3 \times 1} & 0_3 \\
-\left[G \hat{P}_{l_k} \times \right]^G g & G_{L_r} \hat{R} \\
0_{3 \times 1} & G_{L_r} \hat{R} \\
-\left[G \hat{P}_{l_1} \times \right]^G g & G_{L_r} \hat{R} \\
-\left[G \hat{P}_{l_1} \times \right]^G g & \cdots \\
\vdots & \vdots \\
-\left[G \hat{P}_{l_1} \times \right]^G g & \vdots \\
-\left[G \hat{P}_{p \times} \right]^G g & \vdots \\
-\left[G \hat{P}_{p \times} \right]^G g & \vdots \\
-\left[G \hat{P}_{p \times} \right]^G g & \vdots \\
-\left[G \hat{P}_{p \times} \right]^G g & \vdots \\
-\left[G \hat{P}_{p \times} \right]^G g & \vdots \\
-\left[G \hat{P}_{p \times} \right]^G g & \vdots \\
-\left[G \hat{P}_{p \times} \right]^G g & \vdots \\
\end{bmatrix}
\]

\[
(75)
\]

Proof. See Appendix F.

Similarly, we can extend this analysis to multiple points, lines and planes. Given that we have $p$ points, $s$ lines and $n$ planes for the estimation, the unobservable directions can be written as:

\[
N_{PLPI} = \begin{bmatrix}
N_{PLPI} & N_{PLPI,2,4} \\
N_{PLPI} & N_{PLPI,2,4} \\
N_{PLPI} & N_{PLPI,2,4} \\
N_{PLPI} & N_{PLPI,2,4} \\
\end{bmatrix}
\]

\[
(76)
\]

where $i \in \{1 \ldots s\}$.

6 Degenerate Cases

In urban structure, it also happens that the intersection of planes and lines are parallel. If in such structured environment, only lines and planes are used for aided INS, given that the unit direction vector of the lines as $G \hat{v}_e$, then the unobservable directions for $l$ parallel lines and $s$ planes (with
parallel intersections) will have one more unobservable direction compared to Case of line and plane. The additional unobservable direction can be written as:

$$N_{LI5} = \begin{bmatrix} 0_{1 \times 3} & 0_{1 \times 3} & G \hat{v}_e^T & 0_{1 \times 3} & 0_{1 \times 4f} & 0_{1 \times 3s} \end{bmatrix}^T$$

(77)

It is obvious that $N_{LI5}$ relates to the line direction (which is parallel to the planes’ intersections).

7 Simulation Results

We have performed extensively Monte Carlo simulations of EKF-based visual-inertial SLAM to validate the proposed CP parameterization for line as well as the observability analysis for aided INS with the combination of geometrical features all in unified CP form. An IMU with a stereo camera moving along a 3D sinusoid trajectory is simulated. The stereo camera produces the projective feature measurements from a pre-generated map. To validate the observability analysis, we have two different EKFs as in [11, 4, 38]: (i) the ideal EKF-based VINS which uses true states
as linearization points, and (ii) the standard EKF-based VINS which uses estimated states for linearization. The ideal EKF should demonstrate consistent estimation, while the standard EKF tends to be overconfident and thus, inconsistent. The normalized estimation error squared (NEES) and root mean squared error (RMSE) are used to evaluate the consistency and accuracy [39] of the estimator. It is clear from Fig. 4 that the standard VINS with line features in proposed 4D CP form can generate accurate estimation results. In addition, as expected, the ideal EKF, which is consistent, performs better than the standard EKF, for the VINS with the combination of geometric features in CP form. Note that by marginalizing features through null space operation [40], we have also extended the standard MSCKF VIO with point features [2] to the case of heterogeneous features and validated its performance in simulations as shown in [30].

8 Conclusions and Future Work

In this paper, we have introduced two sets of unified representations for different geometrical features (points, lines and planes): the quaternion form and the CP form. We particularly advocate a novel CP model to parameterize line features, of which the error states are directly in the minimal 4D Euclidean vector space. Moreover, based on our recent work [1], we have provided an extensive observability analysis together with all the analytically computed Jacobians for aided INS with geometrical features in unified representations, showing that both representations preserve the same observability properties. In particular, the proposed unified CP representation and the observability analysis are validated through extensive Monte-Carlo simulations of vision-aided INS. In the future, we will investigate the (stochastic) observability of aided INS under adversarial attacks [42] or unknown inputs [43] in order to design secure estimators for robot navigation. In addition, based on the quaternion parameterization, we can extend this representation in $SO(3)$ form, and build visual-inertial SLAM system to compare these representations for different geometrical features.
Appendix A: Spherical Form for Line

The relationship between Model 2, 3 and 5 can be described as:

\[ \mathbf{R}(q_l) = [n_e \ v_e] \mathbf{R}_e \]  \hspace{1cm} (78)

where we have:

\[ n_e = \begin{bmatrix} \sin \theta_l & -\cos \theta_l & -\cos \theta_l \sin \phi_l \\ -\cos \theta_l & \cos \alpha_l & \sin \theta_l \sin \phi_l \\ 0 & 0 & \cos \phi_l \end{bmatrix} \]  \hspace{1cm} (79)

\[ v_e = \begin{bmatrix} \sin \theta_l & -\cos \theta_l & -\cos \theta_l \sin \phi_l \\ -\cos \theta_l & \cos \alpha_l & \sin \theta_l \sin \phi_l \\ 0 & 0 & \cos \phi_l \end{bmatrix} \]  \hspace{1cm} (80)

\[ [n_e] v_e = \begin{bmatrix} -\cos \theta_l \cos \phi_l \\ -\sin \theta_l \cos \phi_l \\ -\sin \phi_l \end{bmatrix} \]  \hspace{1cm} (81)

Appendix B: Proof of 4.2

The \( \Gamma_{ij} \) can be written as:

\[ \Gamma_{11} = (G_d \pi [G_{n_e}]^T \mathbf{R} - [[G_{p_k}] [G_{v_e}]^T \mathbf{R}] \Phi_{I11} + [G_{v_e}] \Phi_{I51} \]  \hspace{1cm} (83)

\[ = (G_d \pi [G_{n_e}]^T \mathbf{R} + [G_{v_e}] [G_{p_k} + G_{v_k} \delta t_k - \frac{1}{2} G_{g_k \delta t_k^2} - G_{p_k}] \mathbf{R} \]  \hspace{1cm} (84)

\[ \Gamma_{12} = [G_{v_k}] R \Phi_{I11} = [G_{v_e}] \mathbf{R} \]  \hspace{1cm} (85)

\[ \Gamma_{13} = [G_{v_k}] \delta t_k \]  \hspace{1cm} (86)

\[ \Gamma_{14} = [G_{v_e}] \Phi_{I54} \]  \hspace{1cm} (87)

\[ \Gamma_{15} = [G_{v_e}] \Phi_{I55} \]  \hspace{1cm} (88)

\[ \Gamma_{16} = 2 (\mathbf{R} - [G_{v_e}] [G_{n_e}]^T \mathbf{R}) \Phi_{I54} + [G_{v_e}] [G_{q_1} \mathbf{I}_3 - [G_{q_1}] + [G_{n_e} [G_{q_1} \mathbf{R}] \]  \hspace{1cm} (89)

\[ \Gamma_{17} = -2 (\mathbf{R} - [G_{v_e}] [G_{n_e}]^T \mathbf{R}) [G_{q_1} \mathbf{R} + [G_{q_1} [G_{q_1} \mathbf{R}] \]  \hspace{1cm} (90)

\[ \Gamma_{18} = -2 (\mathbf{R} - [G_{v_e}] [G_{n_e}]^T \mathbf{R}) [G_{q_1} [G_{q_1} \mathbf{R}] \]  \hspace{1cm} (91)
Appendix C: Proof of Lemma 2

\[ \Gamma_{\pi ij} \text{ can be written as:} \]

\[
\Gamma_{\pi 11} = [G_{\pi} n_{ij} R_{\Phi_{ij1}} = [G_{\pi} n_{ij} R_{\Phi_{ij1}}]
\]

\[
\Gamma_{\pi 21} = -G_{\pi}^T \Phi_{ij1} = -G_{\pi}^T [G_{\pi} p_{ij1} + G_{\pi} v_{ij1} \delta t_k - \frac{1}{2} G_{\pi}^2 \delta t^2_k - G_{\pi} p_{ij1} R_{\Phi_{ij1}}]
\]

\[
\Gamma_{\pi 12} = [G_{\pi} n_{ij} R_{\Phi_{ij12}}
\]

\[
\Gamma_{\pi 22} = -G_{\pi}^T \Phi_{ij2}
\]

\[
\Gamma_{\pi 13} = 0
\]

\[
\Gamma_{\pi 14} = 0
\]

\[
\Gamma_{\pi 24} = -G_{\pi}^T \Phi_{ij4}
\]

\[
\Gamma_{\pi 15} = 0
\]

\[
\Gamma_{\pi 25} = -G_{\pi}^T
\]

\[
\Gamma_{\pi 16} = \frac{1}{G_{\pi}} (I_3 - G_{\pi} G_{\pi})
\]

\[
\Gamma_{\pi 26} = -\frac{1}{G_{\pi}} p_{ijk}^T (I_3 - G_{\pi} G_{\pi}) + G_{\pi}^T
\]

Appendix D: Point, Line and Plane in Quaternion

Unit quaternion is widely used for rotation representation, and recently, it is also used for point, line and plane representation [34][20][29]. Except for Kottas et al. [20] has studied the observability for line feature with quaternion representation, no such work on point and plane representation in quaternion.

D.1: Quaternion Error State

A unit quaternion can be written as:

\[
\bar{q} = \begin{bmatrix} q_v \\ q_4 \end{bmatrix}
\]

Following the JPL notation, the error states can be written as:

\[
\bar{q} = \bar{q} + \bar{q} = \delta \bar{q} \otimes \bar{q} = \begin{bmatrix} \frac{1}{2} \delta \theta \\ 1 \end{bmatrix} \otimes \begin{bmatrix} q_v \\ q_4 \end{bmatrix} = \begin{bmatrix} \dot{q}_4 I_3 + |\dot{q}| \\ -\dot{q}^T \end{bmatrix} \begin{bmatrix} q_v \\ q_4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \delta \theta \\ 0 \end{bmatrix}
\]

Therefore, we can get the linearized relationship between the \( \bar{q} \) and \( \delta \theta \) as:

\[
\bar{q} = \delta \bar{q} \otimes \bar{q} - \bar{q} = \begin{bmatrix} \dot{q}_4 I_3 + |\dot{q}| \\ -\dot{q}^T \end{bmatrix} \delta \theta
\]

\[
\bar{q} = \frac{1}{2} \begin{bmatrix} \dot{q}_4 I_3 + |\dot{q}| \\ -\dot{q}^T \end{bmatrix} \delta \theta
\]

\[
\delta q = (\bar{q} + \bar{q}) \otimes \bar{q}^{-1} = \begin{bmatrix} 0_{3x1} \\ 1 \end{bmatrix} + \begin{bmatrix} \dot{q}_4 I_3 - |\dot{q}| \\ q^T \end{bmatrix} \frac{1}{2} \delta \theta
\]

\[
\delta \theta = 2 \left[ \dot{q}_4 I_3 - |\dot{q}| \\ -\dot{q} \right] \bar{q}
\]

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D.2: Point in Quaternion

For the point \( \mathbf{p}_f = r_f \mathbf{b}_f \), we can write it in a unit quaternion form as:

\[
\bar{q}_f = \frac{1}{\sqrt{1 + r_f^2}} \begin{bmatrix} b_f \\ q_f \end{bmatrix}
\]  \hspace{1cm} (116)

The measurement Jacobians can be written as:

\[
H_x = \frac{\partial z}{\partial x} = \begin{bmatrix} \frac{\partial z}{\partial x_f} & \frac{\partial z}{\partial \delta \theta_f} \end{bmatrix}
\]  \hspace{1cm} (117)

The Jacobians w.r.t. the IMU state can be written as:

\[
\frac{\partial z}{\partial I_{\tilde{p}_f}} = H_{\text{proj}}
\]

where:

\[
\frac{\partial z}{\partial I_{\tilde{p}_f}} = \frac{\partial z}{\partial I_{\tilde{p}_f}} (120)
\]

\[
\frac{\partial I_{\tilde{p}_f}}{\partial \delta \theta_f} = \frac{1}{2} \left( G_{q_f}{\tilde{I}_3} + \frac{G_{q_f}}{G_{q_f}} \right)
\]

The Jacobians w.r.t. the point feature in quaternion error states can be written as:

\[
\frac{\partial z}{\partial \delta \theta_f} = \frac{\partial z}{\partial I_{\tilde{p}_f}} \frac{\partial I_{\tilde{p}_f}}{\partial G_{\tilde{p}_f}} \frac{\partial G_{\tilde{p}_f}}{\partial G_{\tilde{q}_f}} \frac{\partial G_{\tilde{q}_f}}{\partial G_{q_f}} (122)
\]

\[
= H_{\text{proj}} \frac{1}{2} \left( (G_{q_f}^2) \frac{1}{G_{q_f}^2} \right) \left( -G_{q_f} G_{q_f}^T \right) \left( G_{q_f} + G_{q_f}^T \right) \left( \frac{1}{2} \left[ G_{q_f} I_3 + \frac{G_{q_f}}{G_{q_f}} \right] \right) (123)
\]

\[
= 1 \left( H_{\text{proj}} G R \left( - (1 + G_{q_f}^2) I_3 + G_{q_f} G_{q_f} - (2 G_{q_f}^2 + 1) \right) \right) (124)
\]

where:

\[
\frac{\partial I_{\tilde{p}_f}}{\partial G_{\tilde{p}_f}} = I_{G R} (125)
\]

\[
\frac{\partial G_{\tilde{p}_f}}{\partial G_{\tilde{q}_f}} = \frac{G_{\tilde{q}_f} I_3}{G_{\tilde{q}_f}} (126)
\]

\[
\frac{\partial G_{\tilde{q}_f}}{\partial G_{q_f}} = \frac{1}{G_{q_f}^2} \left( -G_{q_f} G_{q_f}^T \right) \left( G_{q_f} + G_{q_f}^T \right) \left( \frac{1}{2} \left[ G_{q_f} I_3 + \frac{G_{q_f}}{G_{q_f}} \right] \right) (127)
\]

\[
\frac{\partial G_{\tilde{q}_f}}{\partial G_{\tilde{q}_f}} = \frac{1}{2} \left( G_{q_f} I_3 + \frac{G_{q_f}}{G_{q_f}} \right) (128)
\]
Follow the observability analysis in [4], we can construct observability matrix as:

\[ M(x)_k = H_{xk} \Phi(k, 1) = H_{\text{proj}_G} \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5 \Gamma_6 \]  

where we have:

\[ \Gamma_1 = [(G p_r - G p_l)]_{I_k}^T R \Phi_{11} - \Phi_{51} \]  

\[ \Gamma_2 = [(G p_r - G p_l)]_{I_k}^T R \Phi_{12} - \Phi_{52} \]  

\[ \Gamma_3 = -I_3 \delta t_k \]  

\[ \Gamma_4 = -\Phi_{54} \]  

\[ \Gamma_5 = -I_3 \]  

\[ \Gamma_6 = \frac{1}{2} \left( -(G r_f^2 + 1) I_3 + G r_f [G b_f] - (2G r_f^2 + 1) [G b_f]^2 \right) \]  

Therefore, we can find the null space of the observability matrix as:

\[ N = \begin{bmatrix} I_C^T R G g & 0_3 \\ 0_{3 \times 1} & 0_3 \\ -[G v_{l1}]^T G g & 0_3 \\ 0_{3 \times 1} & 0_3 \\ -[G p_{l1}]^T G g & -I_3 \\ \frac{1}{1 + G r_f^2} \left( [G b_f]^2 - G r_f [G b_f] \right) G g & \frac{2}{1 + G r_f^2} \left( I_3 + \frac{1}{G r_f} [G b_f] + 2 [G b_f]^2 \right) \end{bmatrix} \]  

D.3: Line in Quaternion

For a line with direction vector \( v_e \) and normal direction of the plane constructed from the origin and the line vector \( n_e \), the distance \( d_l \) can be written as:

\[ R(q_l) = [n_e \ v_e \ |n_e|v_e] \]  

Therefore, the line can be written as \( x_l = [q_l^T \ d_l]^T \), therefore, the error states can be written as \( \tilde{x}_l = [\delta \theta_l^T \ d_l] \). Then, the measurement Jacobians can be written as:

\[ H_x = \frac{\partial \tilde{z}_l}{\partial x} = \begin{bmatrix} \frac{\partial \tilde{z}_l}{\partial \tilde{x}_l} & \frac{\partial \tilde{z}_l}{\partial \delta \theta_l} & \frac{\partial \tilde{z}_l}{\partial d_l} \end{bmatrix} \]  

The measurement Jacobians w.r.t. the IMU state as:

\[ \frac{\partial \tilde{z}_l}{\partial \tilde{x}_l} = \frac{\partial \tilde{z}_l}{\partial \tilde{l}} \frac{\partial \tilde{l}}{\partial [n_e \ |n_e|^T]} \frac{\partial [n_e \ |n_e|^T]}{\partial \tilde{x}_l} \]  

\[ = \frac{\partial \tilde{z}_l}{\partial \tilde{l}} \left[ K \ 0_3 \right] \left[ G d_l [I_C^T R G n_e] - [I_C^T R G p_l]^T G v_e \right] 0_{3 \times 9} \left[ I_C^T R G v_e \right] 0_{3 \times 9} 0_3 \]
where we have:

\[
\frac{\partial \tilde{z}_l}{\partial \tilde{l}} = [K \ 0_3] \tag{144}
\]

\[
\frac{\partial \tilde{I}}{\partial \tilde{l}} = \begin{bmatrix}
I_{ne}^{dl} \\
I_{ve}^{dl}
\end{bmatrix} \tag{145}
\]

\[
\frac{\partial \tilde{I}_{ne}^{dl}}{\partial x_I} = \begin{bmatrix}
G_{dl}[G_R]_{ne}^{dl} - \begin{bmatrix} I_{G_R}^{dl} & G_{dl} \end{bmatrix}G_{ve} \\
0_{3 \times 9} & I_{G_R}^{dl} G_{ve}
\end{bmatrix} 0_{3 \times 1} \tag{146}
\]

The measurement Jacobians w.r.t. the line error state as:

\[
\frac{\partial \tilde{z}_l}{\partial \tilde{l}} = \frac{\partial \tilde{z}_l}{\partial \delta \theta_l} \begin{bmatrix}
I_{ne}^{dl} \\
I_{ve}^{dl}
\end{bmatrix} \begin{bmatrix}
\frac{\partial \tilde{I}_{ne}^{dl}}{\partial \delta \theta_l} \\
\frac{\partial \tilde{I}_{ve}^{dl}}{\partial \delta \theta_l}
\end{bmatrix} \frac{\partial \tilde{I}_{ve}^{dl}}{\partial \delta \theta_l} \tag{147}
\]

\[
= \frac{\partial \tilde{z}_l}{\partial \tilde{l}} \begin{bmatrix}
K \ 0_3
\end{bmatrix} \begin{bmatrix}
G_{dl}[G_R]_{ne}^{dl} - \begin{bmatrix} I_{G_R}^{dl} & G_{dl} \end{bmatrix}G_{ve} \\
0_{3 \times 9} & I_{G_R}^{dl} G_{ve}
\end{bmatrix} 0_{3 \times 1} \tag{148}
\]

where we have:

\[
\frac{\partial \tilde{I}_{ne}^{dl}}{\partial \delta \theta_l} = \begin{bmatrix} I_{G_R}^{dl} & -I_{G_R}^{dl} \end{bmatrix} \begin{bmatrix} G_{dl}[G_R]_{ne}^{dl} - \begin{bmatrix} I_{G_R}^{dl} & G_{dl} \end{bmatrix}G_{ve} \\
0_{3 \times 9} & I_{G_R}^{dl} G_{ve}
\end{bmatrix} 0_{3 \times 1} \tag{149}
\]

\[
\frac{\partial \tilde{I}_{ve}^{dl}}{\partial \delta \theta_l} = \begin{bmatrix} G_{dl}[G_R]_{ve}^{dl} \\
0_{3 \times 1}
\end{bmatrix} 0_{3 \times 1} \tag{150}
\]

Following the observability analysis in [4], we can have the observability matrix as:

\[
M_k = H_x \Phi(k, 1) \tag{151}
\]

\[
= \frac{\partial \tilde{z}_l}{\partial \tilde{l}} \begin{bmatrix}
K \ 0_3
\end{bmatrix} \begin{bmatrix}
\Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \Gamma_{15} & \Gamma_{16} & \Gamma_{17} \\
\Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & \Gamma_{25} & \Gamma_{26} & \Gamma_{27}
\end{bmatrix} \tag{152}
\]
where we have:

\[
\Gamma_{11} = \left( \hat{G}_d \left[ G_n \right] \hat{G}_R - \left[ \hat{G}_p \right] \hat{G}_v \right) \Phi_{11} + [G_v] \Phi_{51}
\]

\[
= \left( \hat{G}_d \left[ G_n \right] \hat{G}_R - \left[ \hat{G}_p \right] \hat{G}_v \right) \Phi_{11} + [G_v] \Phi_{51}
\]

\[
\Gamma_{21} = \left[ G_v \right] \hat{G}_I R \Phi_{11} = \left[ G_v \right] \hat{G}_I R
\]

\[
\Gamma_{12} = \left( \hat{G}_d \left[ G_n \right] \hat{G}_R - \left[ \hat{G}_p \right] \hat{G}_v \right) \Phi_{12} + [G_v] \Phi_{52}
\]

\[
\Gamma_{22} = \left[ G_v \right] \hat{G}_I R \Phi_{12}
\]

\[
\Gamma_{13} = \left[ G_v \right] \delta t_k
\]

\[
\Gamma_{23} = 0
\]

\[
\Gamma_{14} = \left[ G_v \right] \Phi_{54}
\]

\[
\Gamma_{24} = 0
\]

\[
\Gamma_{15} = \left[ G_v \right]
\]

\[
\Gamma_{25} = 0
\]

\[
\Gamma_{16} = \hat{G}_d \left[ G_n \right] - \left[ \hat{G}_p \right] [G_v]
\]

\[
\Gamma_{26} = \left[ G_v \right]
\]

\[
\Gamma_{17} = G_n e
\]

\[
\Gamma_{27} = 0_{3 \times 1}
\]

Therefore, we can find the null space as:

\[
N = \begin{bmatrix}
I & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} \\
0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & G_v \\
-\hat{G}_v & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} \\
-\hat{G}_p & G_n e & G_v e & [G_n] G_v & 0_{3 \times 1} \\
-\hat{G}_g & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1}
\end{bmatrix}
\]

(168)

D.4: Plane in Quaternion

Given the a point \( p_f \) in a plane with normal \( n_\pi \) and distance \( d_\pi \), we have:

\[
p_f^T n_\pi - d_\pi = 0
\]

(169)

Then, the plane in quaternion can be written as:

\[
\tilde{q}_\pi = \frac{1}{\sqrt{1 + d^2_\pi}} \begin{bmatrix} n_\pi \end{bmatrix} = \begin{bmatrix} q_\pi \end{bmatrix}
\]

(170)

Then, the measurement Jacobians can be written as:

\[
H_x = \frac{\partial \tilde{z}_\pi}{\partial x} = \begin{bmatrix} \frac{\partial \tilde{z}_e}{\partial x} & \frac{\partial \tilde{z}_e}{\partial \theta_\pi} \end{bmatrix}
\]

(171)
The measurement Jacobians w.r.t. the IMU state can be written as:

\[
\frac{\partial \tilde{x}_\pi}{\partial \dot{t} q_{\pi}} = \frac{\partial \tilde{x}_\pi}{\partial \dot{t} \tilde{q}_{\pi}} \frac{\partial \left[ \begin{array}{c} \dot{t} \tilde{n}_{\pi} \\ \dot{t} \tilde{d}_{\pi} \end{array} \right]}{\partial \dot{t} \tilde{q}_{\pi}} (172)
\]

\[
= 2 \left[ \dot{t} q_{\pi} \mathbf{I}_3 - \dot{t} \tilde{q}_{\pi} \right] \frac{1}{(1 + \dot{t} d_{\pi}^2)^2} \left[ \begin{array}{c} (1 + \dot{t} d_{\pi}^2) \mathbf{I}_3 - \dot{t} d_{\pi} \dot{t} n_{\pi} \\ 0_{1 \times 3} \end{array} \right] \left[ \begin{array}{c} \dot{t} \mathbf{R}_G^{T} n_{\pi} \\ 0_{3 \times 9} \end{array} \right] (173)
\]

where we have:

\[
\frac{\partial \tilde{x}_\pi}{\partial \dot{q}_{\pi}} = 2 \left[ \dot{t} q_{\pi} \mathbf{I}_3 - \dot{t} \tilde{q}_{\pi} \right] (174)
\]

\[
\frac{\partial \dot{t} q_{\pi}}{\partial \dot{t} \tilde{q}_{\pi}} = \frac{1}{(1 + \dot{t} d_{\pi}^2)^2} \left[ \begin{array}{c} (1 + \dot{t} d_{\pi}^2) \mathbf{I}_3 - \dot{t} d_{\pi} \dot{t} n_{\pi} \\ 0_{1 \times 3} \end{array} \right] (175)
\]

\[
\frac{\partial \dot{t} \tilde{n}_{\pi}}{\partial \dot{t} \tilde{q}_{\pi}} = \left[ \begin{array}{c} \dot{t} \mathbf{R}_G^{T} n_{\pi} \\ 0_{3 \times 9} \end{array} \right] (176)
\]

The measurement Jacobians w.r.t. the plane quaternion error states can be written as:

\[
\frac{\partial \tilde{x}_\pi}{\partial \dot{q}_\pi} = \frac{\partial \tilde{x}_\pi}{\partial \dot{q}_\pi} \frac{\partial \left[ \begin{array}{c} \tilde{n}_{\pi} \\ \tilde{d}_{\pi} \end{array} \right]}{\partial \dot{q}_\pi} \frac{\partial \left[ \begin{array}{c} \tilde{n}_{\pi} \\ \tilde{d}_{\pi} \end{array} \right]}{\partial \dot{q}_\pi} (177)
\]

\[
= 2 \left[ \dot{t} q_{\pi} \mathbf{I}_3 - \dot{t} \tilde{q}_{\pi} \right] \frac{1}{(1 + \dot{t} d_{\pi}^2)^2} \left[ \begin{array}{c} (1 + \dot{t} d_{\pi}^2) \mathbf{I}_3 - \dot{t} d_{\pi} \dot{t} n_{\pi} \\ 0_{1 \times 3} \end{array} \right] \left[ \begin{array}{c} \dot{t} \mathbf{R}_G^{T} p_{\pi} \mathbf{I}_3 \\ 0_{3 \times 1} \end{array} \right] (178)
\]

\[
= 2 \left[ \dot{t} q_{\pi} \mathbf{I}_3 - \dot{t} \tilde{q}_{\pi} \right] \frac{1}{(1 + \dot{t} d_{\pi}^2)^2} \left[ \begin{array}{c} (1 + \dot{t} d_{\pi}^2) \mathbf{I}_3 - \dot{t} d_{\pi} \dot{t} n_{\pi} \\ 0_{1 \times 3} \end{array} \right] \left[ \begin{array}{c} \dot{t} \mathbf{R}_G^{T} p_{\pi} \mathbf{I}_3 \\ 0_{3 \times 1} \end{array} \right] (179)
\]

where we have:

\[
\frac{\partial G}{\partial \dot{q}_\pi} = \left[ \begin{array}{c} \dot{t} \mathbf{R}_G^{T} p_{\pi} \mathbf{I}_3 \\ 0_{3 \times 1} \end{array} \right] (180)
\]

\[
\frac{\partial G}{\partial \dot{q}_\pi} = \left[ \begin{array}{c} \dot{t} \mathbf{R}_G^{T} p_{\pi} \mathbf{I}_3 \\ 0_{3 \times 1} \end{array} \right] (181)
\]

\[
\frac{\partial G}{\partial \dot{q}_\pi} = \left[ \begin{array}{c} \dot{t} \mathbf{R}_G^{T} p_{\pi} \mathbf{I}_3 \\ 0_{3 \times 1} \end{array} \right] (182)
\]

\[
\frac{\partial G}{\partial \dot{q}_\pi} = \left[ \begin{array}{c} \dot{t} \mathbf{R}_G^{T} p_{\pi} \mathbf{I}_3 \\ 0_{3 \times 1} \end{array} \right] (183)
\]

\[
\frac{\partial G}{\partial \dot{q}_\pi} = \left[ \begin{array}{c} \dot{t} \mathbf{R}_G^{T} p_{\pi} \mathbf{I}_3 \\ 0_{3 \times 1} \end{array} \right] (184)
\]

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Following the observability analysis in [4], we can construct the observability matrix as:

\[
M_k = H_k \Phi(k, 1)
\]

\[
= \frac{\partial \tilde{z}_e}{\partial \delta t_{k}} \begin{bmatrix}
\frac{\partial \tilde{\delta}_{k}}{\partial \delta t_{k}} & 0_{1 \times 3} & 1
\end{bmatrix}
\begin{bmatrix}
\Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \Gamma_{15} & \Gamma_{16} \\
\Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & \Gamma_{25} & \Gamma_{26}
\end{bmatrix}
\]

(185)

(186)

where:

\[
\Gamma_{11} = [^G G n_\pi ]_l \Phi_{11} = [^G G n_\pi ]_l \Phi
\]

(187)

\[
\Gamma_{21} = -[^G G n_\pi]_l [^G p_{l_1} + ^G v_{I_1} \delta t_k - \frac{1}{2} G \delta t_k^2 - ^G p_{l_k} ]_l \Phi
\]

(188)

\[
\Gamma_{12} = [^G G n_\pi]_l \Phi_{12}
\]

(189)

\[
\Gamma_{22} = -[^G G n_\pi]_l \Phi_{22}
\]

(190)

\[
\Gamma_{13} = 0_3
\]

(191)

\[
\Gamma_{23} = -[^G G n_\pi]_l \delta t_k
\]

(192)

\[
\Gamma_{14} = 0_3
\]

(193)

\[
\Gamma_{24} = -[^G G n_\pi]_l \Phi_{54}
\]

(194)

\[
\Gamma_{15} = 0_3
\]

(195)

\[
\Gamma_{25} = -[^G G n_\pi]_l
\]

(196)

\[
\Gamma_{16} = \frac{1}{2}
\]

(197)

\[
\Gamma_{26} = -\frac{1}{2}
\]

(198)

Therefore, we can find the null space for the observability matrix as:

\[
N = \begin{bmatrix}
[^G G h]_l \Phi & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} \\
0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} \\
-[^G v_{I_1}]_l [^G G g] & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} \\
0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} \\
-[^G p_{l_1}]_l [^G G g] & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} \\
-2 \left( I_3 + \frac{1}{2} [^G G n_\pi]_l + \frac{1}{2} [^G G n_\pi]_l^2 - [^G G n_\pi]_l G \right) [^G G g] & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1}
\end{bmatrix}
\]

(199)

**Appendix E: Proof of Lemma 3**

First, it is straightforward to verify the null space \(N_{l_{\pi 1}}\) related to the rotation around the gravity direction. If the line is parallel to the plane, we have the line direction vector \(^G v_L\) is perpendicular to the plane normal direction \(^G n_{\Pi}\), that is \(^G v_L^T G n_{\Pi} = 0\), hence we have:

\[
H_{l_k} \Phi(k, 1) N_{l_{\pi 5}} = \begin{bmatrix}
H_{l_k} K_{l_k} R & 0_{3 \times 1} \\
0_{3 \times 1} & R
\end{bmatrix} \begin{bmatrix}
[^G v_{l_k}]_l [^G G G v_{l_k}]_l \delta t_k = 0
\end{bmatrix}
\]

(200)

Therefore, if the line is parallel to the plane, the system will have one more unobservable direction \(N_{l_{\pi 5}}\). The last is to verify the \(N_{l_{\pi 2:4}}\). For simplicity, we write (69) as:

\[
H_{l_k} \Phi(k, 1) = \begin{bmatrix}
H_{l_k} \Phi(k, 1) \\
H_{l_k} \Phi(k, 1)
\end{bmatrix}
\]

(201)
where $H_{I_k}^I$ and $H_{I_k}^P$ are the Jacobians w.r.t. the IMU state and the line and plane features. It is easy to verify that:

$$
H_{I_k}^I \Phi(k,1)N_{l\pi 2:4} = 0 \tag{202}
$$

$$
H_{I_k}^P \Phi(k,1)N_{l\pi 2:4} \left( G_{\Pi}^{\hat{R}} G_{L}^{\hat{R}} \right)^\top = 0 \tag{203}
$$

Since $G_{\Pi}^{\hat{R}} G_{L}^{\hat{R}}$ is a rotation matrix, it is of full rank. Therefore, by multiplying $G_{\Pi}^{\hat{R}} G_{L}^{\hat{R}}$ to both side of Eq. (203), we can arrive at:

$$
H_{I_k}^P \Phi(k,1)N_{l\pi 2:4} = 0 \tag{204}
$$

Therefore, we get $H_{I_k}^I \Phi(k,1)N_{l\pi 2:4} = 0$.

**Appendix F: Proof of Lemma 4**

It is not difficult to verify the null space $N_{pl\pi 1}$ related to the rotation around the gravity direction. Then we need to verify the $N_{pl\pi 2:4}$. For simplicity, we write (205) as:

$$
H_{I_k} \Phi(k,1) = \begin{bmatrix}
H_{I_k}^I \Phi(k,1) \\
H_{I_k}^P \Phi(k,1) \\
H_{I_k}^\pi \Phi(k,1)
\end{bmatrix} \tag{205}
$$

where $H_{I_k}^I$, $H_{I_k}^P$, and $H_{I_k}^\pi$ are the Jacobians w.r.t. the IMU state together with the point, line and plane feature, respectively. It is easy to verify that:

$$
H_{I_k}^I \Phi(k,1)N_{pl\pi 2:4} = 0 \tag{206}
$$

$$
H_{I_k}^P \Phi(k,1)N_{pl\pi 2:4} = 0 \tag{207}
$$

$$
H_{I_k}^\pi \Phi(k,1)N_{pl\pi 2:4} \left( G_{\Pi}^{\hat{R}} G_{L}^{\hat{R}} \right)^\top = 0 \tag{208}
$$

Similarly, since $G_{\Pi}^{\hat{R}} G_{L}^{\hat{R}}$ is of full rank, we can multiply both side of Eq. (208) with $G_{\Pi}^{\hat{R}} G_{L}^{\hat{R}}$ and arrive at:

$$
H_{I_k}^\pi \Phi(k,1)N_{pl\pi 2:4} = 0 \tag{209}
$$

Therefore, we get $H_{I_k} \Phi(k,1)N_{pl\pi 2:4} = 0$.

**References**


[28] Adrien Bartoli and Peter Sturm. “Structure From Motion Using Lines: Representation, Triangulation and Bundle Adjustment”. In: Computer Vision and Image Understanding 100.3 (2005), pp. 416–441.


