Mathematical play and playful mathematics: A guide for early education

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This chapter is about the role of mathematics in children's play and the role of play in early mathematics education. The confluence of environment and biology guarantees that virtually all children acquire major aspects of everyday mathematics (EM). Children's EM is ubiquitous, often competent, and more complex than usually assumed. It involves activities as diverse as perceiving which of two plates of cookies has "more" and reflecting on the issue of what is the largest number. It should therefore come as no surprise that EM is a significant aspect of children's play. Children use informal skills and ideas relating to number, shape, and pattern as they play with blocks or read storybooks. Indeed, EM provides the cognitive foundation for a good deal of play, as well as for other aspects of the child's life. Even more remarkably, spontaneous play may entail explicit mathematical content: young children can enjoy explorations of number and pattern as much as messing around with clay. Further, children also play with the teacher's mathematics—the lessons taught in school. Research on EM provides guidelines for the goals, content, and nature of early childhood mathematics education. If we create and employ a challenging and playful mathematics curriculum, then, as the title of this book suggests, play can indeed produce learning—even mathematics learning..

Children's Everyday Mathematics: What It Is and Why All Children Develop It

Everyday mathematics (EM) refers to what Dewey (1976) in the early part of the 20th century called the child's "... crude impulses in counting, measuring, and arranging things in rhythmic series..." (p. 282). Vygotsky (1978) also pointed to the phenomenon: "... children's learning begins long before they enter school... they have had to deal with operations of division, addition, subtraction, and the determination of size. Consequently, children have their own preschool arithmetic, which only myopic psychologists could ignore" (p. 84). Vygotsky's reference to arithmetic did not of course imply that children's EM is the written, algorithmic arithmetic of the schools. EM seldom takes written form and does not involve conventional procedures for adding or dividing. Instead, EM entails such activities as determining how some cookies should be fairly divided among siblings or judging that adding a toy to a given collection results in "more," which children definitely prefer to "less." Upon a little reflection, every parent realizes that children can do things like this and can be said to possess an EM. But the nature and extent of children's EM are poorly understood.

All children develop some form of EM. It is a fundamental category of mind and is as natural and ubiquitous as crawling. From birth, children in all cultures develop in physical environments containing a multitude of objects and events that can support mathematics learning in everyday life (Ginsburg & Seo, 1999). A large number of parallel bars is on the side of babies' cribs; stalks of corn in a field are similarly arranged in rows; there is a larger number of candies or stones in one collection than other; the toy is under the chair, not on top of it; blocks can be cubes and balls are spheres; in the field, one cow is front of the tree and another behind it. Although varying in many ways, including the availability of books, schools and "educational" toys, all environments surely contain objects to count, shapes to discriminate, and locations to identify. The objects and events are not themselves mathematics, but they afford mathematical thinking. The chicks wander around the farmyard. They are chicks, not numbers, but they can be counted. Bars on a crib are pieces of wood, but they can be seen as parallel lines. The chicks and the bars can be the food for mathematical thought. In brief, children are universally provided with common “supporting environments” for at least some aspects of mathematical development (Gelman, Massey, & McManus, 1991, p. 254).

Of course, the existence of mathematical food for thought does not guarantee that it will be ingested, let alone digested. Yet several factors guarantee that in virtually all children take advantage of the environmental opportunity and develop key features of everyday mathematical knowledge. First, as Piaget (1952b) maintained, “general heredity,” a kind of instinct to learn, insures that all children “adapt” to their environments, attempting to make sense of them. This thought is echoed in recent theory: “… we can think of young children as self-monitoring learning machines who are inclined to learn on the fly, even when they are not in school and regardless of whether they are with adults” (Gelman, 2000, p. 26). Because mathematical thinking is required to make sense of the universal environment, the self-monitoring learning machines tend to learn some mathematical ideas. Children *invent* their own addition methods in the absence of adult instruction (Groen & Resnick, 1977). For example, at first, virtually all children add by “counting all.” Shown that Johnny has two apples, and Sally three, 4-year-olds commonly determine the sum by counting, “One, two, three…four, five.” After using this method for a period of time, children then tend to invent for themselves the more economical method of “counting on from the larger number.” Given the same problem, children often begin with the larger set, saying, “three… four, five.” It is fortunate that children can learn on their own because most parents are unaware of many features of EM, just as psychologists and educators were until the advent of contemporary research on the topic. Thus, in first half of the 20^{th} century, many educational theorists assumed that “young children started school with no prior mathematical knowledge or experience and that limited instruction” was sufficient for the early grades (Balfanz, 1999, p. 8).

Second, children are either biologically endowed with specific mathematical concepts or biologically primed to learn them. For example, Gelman (2000) proposes that, "… we are born with number-relevant mental structures that promote the development of principles for counting" (p. 36). Similarly, Geary (1996) argues that all children, regardless of background and culture, are endowed with "biologically primary" abilities including not only number, but also basic geometry. These kinds of abilities are universal to the species (except perhaps for some retarded or otherwise handicapped children) and require only a minimum of environmental support to develop. The evidence for claims like these is mainly of two types. One is that some mathematical concepts seem to emerge very early in infancy and even in animals. Thus, Spelke (2003) shows that 6-month old infants can discriminate between collections of 6 and 12 dots, and reviews other studies showing that "… capacities to discriminate between numerosities have been found in nearly every animal tested, from fish to pigeons to rats to primates…" (p. 286). The other evidence for the claim is that many everyday mathematical concepts appear to be universal (Klein & Starkey, 1988).

Third, it is often useful to learn mathematical concepts. If you are inadequately nourished, you need to choose more food, rather than less; and if you are gluttonously nourished, you need to choose less food rather than more. If you want to outshine your peers, you need to learn about length ("my tower is higher than yours") and equality ("I have as much as you"). And if you want to buy something, you need to understand and count your money, even in different denominations.

Fourth, the social environment also contributes to the development of mathematical knowledge. Almost all cultures offer children at least one basic mathematical concept and tool—namely counting. Even groups lacking formal education have developed elaborate counting systems (Zaslavsky, 1973). In many cultures, parents engage in various *informal* activities designed to promote mathematics learning. They count with their children or read books about numbers or shapes. Parents also play mathematics related board and card games with their children (Saxe, Guberman, & Gearhart, 1987).

Fifth, we use various methods to teach young children mathematics. In some cultures, television shows like *Sesame Street*, computer programs and various toys make elementary mathematics available to young children. Many U.S. states now mandate preschool mathematics instruction and some schools use mathematics curricula (e. g., Griffin, 2004; Sophian, 2004). Research shows that children can be taught more mathematics than commonly assumed, including symmetry (Zvonkin, 1992) and spatial relations, among other topics (Greenes, 1999).

This confluence of environment and biology guarantees that mathematical ideas of number, space, geometry and the like are essential parts of children's (and adults') cognitive apparatus. Indeed, EM is so fundamental and familiar that we seldom think of it as mathematics. But knowing that sizes can differ in orderly ways enables the child to select the longer block to create the higher tower. It also enables the child to understand that papa bear is bigger than mama bear who in turn is bigger than baby bear. Further, a simple idea of co-variation (Nunes & Bryant, 1996) allows the child to understand that the bears' sizes are directly related to their beds': papa bear gets the biggest bed, mamma bear the next biggest, and of course baby bear the smallest. It is hard to see how children or adults could survive in the ordinary environment without basic intuitions of more, less, near, far, and the like. Intuitions like these are so essential to human survival that they may well be universal (Klein & Starkey, 1988). Virtually all preschool children can be expected to employ fundamental mathematical ideas.

In brief, all normal children have the capacity, opportunity, and motive to acquire basic mathematical knowledge. It should come as no surprise then to learn that EM has a key role in play.

Several Kinds of Mathematical Play

Consider three types of mathematical play. EM is deeply embedded within play; play may center on mathematical ideas and objects; and play may center on the mathematics that the teacher has taught.

Mathematics Embedded In Play

EM As The Foundation for Reading

EM manifests itself in many unexpected activities, one of them being make-believe reading. Preschool children sometimes go to the reading area, select a book, perhaps one the teacher has recently read to the group, and play at "reading" it. Of course, they can seldom sound out or recognize individual words. Instead, they try to construct a story that resembles what they remember from story time or makes sense of the picture on the page.

Here is an example taken from videotaped observations of children's everyday, unscripted behavior in a daycare center. Jessica brings a book to a table where Matthew and Ralph are sitting side by side. Jessica and Ralph are 5 and Matthew is 4. They are all from low-income families, attending a publicly supported day care center. Jessica sits around the corner of the table from Matthew and Ralph. She pretends that she is the teacher and that the lesson is reading; Matthew and Ralph pretend that they are students. She opens the book, picks it up, holds it in her right hand, and tries to show a page to Matthew and Ralph. Before she can say anything, Ralph says to her, "You can't read it like that. You can't see it" (Ginsburg & Seo, 2000).

Ralph's comment reveals at least two important kinds of thinking. First, he is able to engage in *perspective taking*. He considers Jessica's orientation in relation to the book. He notices that from her point of view the book is held at a bad angle and more or less upside down. Second, he knows that it is very hard to read pages from such a perspective. Jessica is responsive to the feedback: she adjusts the orientation of book so that all of them can see it. "I can see," Ralph says. "Me too," Matthew says.

This is shows how EM—understanding something about orientation, perspective, and angle—is a basic component of good "pre-literacy." Children need to learn that reading requires viewing the book in the right orientation at a reasonable angle!

Next, as Matthew stands close to Ralph to see the book, Ralph seems annoyed and says to him, "Sit down, Matthew." Matthew returns to his seat. But from there he cannot see clearly the pictures on the book. He moves his chair closer to Ralph's, saying, "Let me get a little bit far." "Little closer," Ralph corrects him. "Little closer, I mean," Matthew says.

This episode shows that as they prepare themselves for the story, Matthew and Ralph spontaneously deal with the idea of *relative distance*. They do not want to keep too close to each other, but they both want to see the book. So they sit side-by-side, not too close, and yet not too far from the book. Furthermore, they attempt to use the proper language to express these EM ideas. Matthew clearly has the idea of moving closer for a better view. He expresses this idea in terms of what an adult would consider an odd construction, "Let me get a little bit far." Apparently, he means that he wants to be "a little bit far" from the book as opposed to "a lot far." Although this makes perfect sense, Ralph corrects him, pointing out essentially that in this situation we usually talk in terms of greater closeness rather than lesser distance. So children try to express EM ideas in words and sometimes learn from each other the desired conventional language.

Jessica then "reads" the book to Matthew and Ralph, making up a story based on the pictures. She comments on a picture of pumpkins on the page, "That's a lot." Matthew wonders how much is "a lot." He stretches out his arms and asks her, "A lot like this bunch?" Jessica nods her head, indicating agreement. He stretches his arms further apart and asks her if that also indicates "a lot," saying, "How about this?" Jessica nods again. Matthew stretches his arms even further and asks, "How about this?" She nods affirmatively yet a third time, indicating that all of the arm gestures indicate "a lot."

Matthew seems to be trying to get a handle on what "a lot" means. He asks whether different "amounts" all indicate a lot. It's like saying, "Is 25 a big number? And 35? And 43 too?" To do this, he has to distinguish among different *relative magnitudes*; he has to know that this arm span is larger than this one, and that the next is even larger still. Further, Mathew attempts to represent an abstract idea—"a lot of pumpkins"—by stretching apart his arms. He enacts the idea with his body.

Jessica continues telling the story: "…Put the masks back into that toy box… And then you can take it back out." Matthew repeats, "Take it back out!" So now the children have shifted from ideas about magnitude to the issue of *location*: put things "into" the box and take things "out" of the box.

The children go on to discuss the degree to which a pumpkin was "this tiny scary." My claim is that understanding a story—almost any story!—requires comprehending EM ideas of magnitude, location, quantity, and the like. The same is true for adults reading Shakespeare's sonnets (Ginsburg & Seo, 1999).

Block play

Children clearly deal with ideas of shape, space and pattern when they play with blocks (Leeb-Lundberg, 1996), as Froebel had intended when he created this "gift" (Brosterman, 1997). But many educators and parents often fail to appreciate that many different kinds of EM make their appearance in block play.

Chris and Jeff, 4-year-old boys from low-income families, are playing in the block area. In the center of the block area there was a huge structure that towered above the two boys. The structure was about a foot taller than Chris. The boys had built it earlier during their "work time." Both children could reach the very top of the building by reaching their arms high above their heads and standing on their toes. The building consists of a series of quadruple unit blocks stacked one on top of the other. Blocks are placed parallel to one another and then two more are placed on the top of those on each end to create a series of square shaped levels up from the floor.

Chris and Jeff are sitting next to the block structure playing with toy people. Jeff says, "I am a boy. I am the strongest boy." To which Chris responds, "I am the strongest boy, too." This competitive concern about relative magnitude continues through most of the segment and eventually seems to bring about a large number of EM activities. For example, Chris soon says, "I can jump very high!" Jeff responds by saying, "I can jump very high than you!"

The children's language carries a tone of one-upmanship; their play stems directly from the desire to say or do something that outweighs what the other just said and did. In a sense, EM forms the cognitive basis for much competition: "I have more x than you." The example also illustrates an important point about early EM language: children's ideas are more advanced than the ability to express them in words. When Jeff says, "I can jump very high than you!" his idea is clear, but his linguistic construction is unorthodox. In this case, at least, thought leads language; language does not facilitate thought.

The boys then begin a competitive game in which the challenge is to make toy people jump from a higher and higher levels of the building. Chris says, "Put them up high, high, high" and reaches up to the top levels of the building to toss his toy people over the edge. He shows some understanding of various heights and the relationship between the distances at which he can place the toy people on the building. When he says, "High, high up," he places the toy people as high as he can.

The teacher comes over to check on what they are doing. Chris says, "We're putting people up there and they're falling." The teacher replies, "Oh my goodness, what a dreadful idea." Jeff says, "They're sleeping when they are falling." The teacher responds, "Oh I see, it's a fantasy, a pretend game."

The teacher entirely missed the nature of the mathematical activity in which the boys were engaging. Instead she moralized about the violent nature of the game, calling the boys' activity "… a dreadful idea." Jeff countered by making the people less than fully conscious, presumably to lessen the impact of the fall. The teacher let him off the hook by calling the game a fantasy—as if the boys actually thought it was real!

Play Centering on Mathematics

Children do not only play with blocks or dress-up clothes or Legos. They play with mathematics directly.

"*We all got one hundred.*" Steven, a low-SES African-American Kindergarten child, is sitting at a round table, playing with very small stringing beads. As he carefully pours the beads in his hand onto the table, Steven considers their number. Instead of saying "many," or "lots of (or a lot of)," like many young children, he says out loud to no one in particular, "Oh, man. I got one hundred." This may be an estimate, an indication of "a lot," or even the biggest number he knows. In any event, he wishes to find out exactly how many he does have and counts to find out. He picks up the beads one by one, and counts, "One, two, three..." When Steven picks out the tenth one, Barbara joins him, "Ten, eleven, twelve…" However, although uttering the number words in sequence, Barbara is not actually enumerating the beads. Instead, she sweeps up beads from the table into her hand. They keep counting. Steven drops the twenty-sixth one, but ignores it and continues counting, picking up one bead and saying "twenty-seven." When he takes time to grab the twenty-seventh one, Barbara keeps pace with him, saying as he does "twenty-seven." Steven drops the thirtieth bead. He pauses for a second and says, "Wait! I made a mistake."

Steven pours the beads in his hands on the table and starts to count them again. He really wants to get it right. "One, two, three…" When he counts "three," Barbara picks out one bead, shows it to him, and says, "I have one." But Steven ignores Barbara's distractions and concentrates all his attention on his counting. When he counts "five," Barbara joins his counting, "five." When he counts "ten," Barbara again shows her beads to Steven, "I got, look…" Steven again ignores her and continues counting. Barbara keeps pace, uttering the same number words. When he counts "twelve," Barbara shouts meaningless words in his ear, as if she wants to distract his attention. Steven ignores her again, and keeps counting, "nineteen, twenty [at twenty, he puts out two beads], twenty-one…" When Steven counts forty-seven, a girl asks, "What do you count?" Again, he ignores her and keeps counting. After the forty-nine, Steven pauses. Interestingly, Barbara, who interrupted his counting by shouting meaningless words in his ear, rescues him from being stuck at forty-nine. As Barbara says, "fifty," Steven follows her, “fifty, fifty-one, fifty-two….”

When they count “fifty-two,” Ruthie comes to the table, picks out one bead, and joins their counting, “fifty-two, fifty-three….” Madonna also comes to the table, tries to find a place at the table, picks out one bead, and joins the counting, “fifty-six, fifty-seven…” The girls’ counting breaks the one-to-one correspondence between number words and beads; the girls sometimes pick out several beads at once or sometimes don’t pick out a single bead, though they correctly say the number words in sequence. They are not engaged in enumeration and instead seem to enjoy the repetitive behaviors of picking up beads and saying the number words in a certain tune and rhythm. Steven does not seem to care about them or what they are doing; he does not exchange a word with them.

After “seventy-nine,” Steven again pauses. As the girls say, “eighty,” Steven continues the counting, “eighty, eighty-one…” When they count “eighty-five,” the girls compete with one another to grab more beads. The plastic container is turned over, and the beads in the container are dropped on the table and the floor, rolling in every direction. The girls grab the beads, trying to get more than one another. Although their fight over the beads leads to chaos, Steven’s persistence is surprising. He keeps counting, “eighty-six, eighty-seven… ninety-four.” He makes several mistakes, but this time does not correct them. He seems to be determined to reach one hundred no matter what.

After he picks up the ninety-fourth bead, he finds no bead on the table. He bends over and picks out a bead from those on the floor, and continues counting, “ninety-five, ninety-six, ninety-seven… After a short moment, Madonna shouts, “One hundred!” raising her arm in triumph. Steven, Barbara, and Ruthie say, “One-hundred!” right after her. And Steven says, “We all got one hundred.” For them, “one hundred” is a special number and needs to be celebrated.

Steven's counting provides an example of when, why and how counting is used in young children's everyday activities, not simply to show how high and how well he can count. At first, it was a tool to solve the mathematical problem of how many beads were on the table. I call this a mathematical problem because there was no utility in knowing the number. The situation did not involve getting more beads than some one else or competing in the creation of the largest number of beads. As Steven engaged in this activity, he seemed to become interested in counting as an activity for its own sake. His play with the beads morphed into play with the counting system itself. He corrected mistakes; he wanted his counting to be done right and well (except toward the end); and he wanted to reach one hundred, a special number. He absorbed himself in counting, ignoring all distractions, and finally reached his goal. In most Kindergarten classrooms, counting from one to 100 is often seen as boring drill and is usually considered to be a difficult task. Indeed, the California Academic Content Standards (California Department of Education, 1998) set 30 as the "developmentally appropriate" highest number to which Kindergarten children should be expected to count. But for Steven, counting to 100 appeared to be enjoyable and yet serious "play."

Play With the Mathematics That Has Been Taught

As they play "teacher," children also play with the mathematics they learn from their teachers. Here are some examples provided by my student, Luzaria Dunatov. She writes:

> *Background. I teach at PS 51, the Bronx New School, a public school of choice. The children are enrolled by lottery and come from different areas of the Bronx. The student population is very ethnically and socio economically diverse. This is my third year teaching. I teach a class of 26 kindergarten students.*

Reading numbers. First thing in the morning, Joanna and Nick are assigned to a literacy "center" called "read around the room." Children in this center take turns pointing with a yardstick to various words scattered around the room and then reading them. They can choose whatever it is they want to read from charts we have created as a class, labels, signs, graphs, and the "word wall," which contains high frequency sight words alphabetically arranged. When I modeled what to do in this center, I used the pointer to point to and read words on the word wall and letters on the ABC chart (which links the letters of the alphabet with pictures). I did not model the "reading" of any number charts or number lines.

On this day, I observe Joanna and Nick standing in front of the classroom calendar. As she points to different numbers on the January calendar, Joan pauses and waits for Nick to say the target number. She does not point to the numbers in any particular order. As she points, Nick correctly reads the numbers 15, 23, 11, 5, 8, 14, 17, 9, 23, 10.

A few minutes later I turn to see what they are up to. They are in front of the 100 chart. I have used the 100 chart in my math lessons and daily during morning meeting. It is a well-known resource in our classroom. Joanna is pointing to the numbers as Nick counts. I see them when they are already at the number 79. She points to the numbers in sequence and Nick keeps up with her pointing as he counts. Joanna says, "Say it louder!" She starts pointing too fast for Nick to keep up with so he falls behind by one number. When Joan points to the number 98, Nick is saying 97. Joan waits and stays at 98 until Nick says 98, then she continues and points to 99 and 100. They walk away to find something else to read.

Keisha and Derek are also walking around the room with a pointer reading things on the walls around the classroom. Derek is pointing and counting on the number line. Keisha watches and walks with him as he counts higher on the number line. Derek pauses after counting and pointing to the number 79. He points to the number 80 and says, "What number is this?" Keisha tells him it's 80. Derek continues to point and count. He gets stuck again after the number 89. He asks Keisha again, "What number is this?" She tells him 90. Derek continues counting once again. At 110, Derek says, "100 and 0." For 111, he says, "100 and 1." Keisha chimes in to continue counting with him. It is interesting to see how Keisha reacts when he gets off track. She identifies his errors, and quickly begins to correct them by counting with him. When Derek needs help counting, she is quick to supply him with the numbers he can't read. She is aware of his abilities and chimes in when he needs extra support. Derek is comfortable asking his classmate for help when he has trouble.

Pattern. Michelle is busy pointing to her stockings during morning meeting. She is wearing striped colorful stockings and she is saying the colors aloud as she points to the stripes. I ask Michelle what she is doing. She says, "I was figuring out a pattern. It keeps going." She says, "Purple, green, pink, blue, orange, white, purple, green pink, blue, orange, white, purple…"

We have done a good amount of work around patterns. The children have created and extended patterns made of colors, pattern blocks, and other math manipulatives. Amazed by all the different kinds and complexity of patterns I was observing, I briefly mentioned how some kids were making ABC patterns and how some where making ABB patterns or AB patterns, etc. I read their patterns to the class using letter notation.

I say, "Wow! That's a long and tricky pattern. I point to each of the stripes and say," A, B, C, D, E, F, A, B, C, D, E, F." I proclaim, "It's an A, B, C, D, E, F, pattern."

Base ten. During choice time, Derek has chosen to play at the math manipulatives center. Derek is using linking cubes to make towers of 10. He already had 3 towers of 10 lying next to each other. He says, "Look, Luzaria, I'm doing by 10's. I can make a pattern like brown, green, or A, B." As he makes the fourth tower of 10, he is measuring it up against the other towers of 10 to see how many more cubes he needs. "I need one more."

Derek is transferring knowledge from our math work during morning meeting where we count the days in school using linking cubes. When we have 10 loose cubes, we snap them together into groups of 10. But he modifies the strategy a bit when, instead of counting 10 cubes and then snapping them together, he compares the tower he is building to the other towers of 10.

Measuring. Joan and Shelly are playing with tape measures at the math manipulatives center. Joan is measuring the bin that holds the tape measures. She holds the tip of the measuring tape on one side of the top of the bin and stretches the tape across the bin. She says, "It's 12, it's 12."

Shelly says, "I got 21."

Joan says, "Let me see," and measures it again. See, you made a mistake. It was 12."

Shelly says, "We measure it around like this." Shelly takes the tape and measures from the bottom of the container up to the top, across the top to the other end, and down to the bottom.

Joan says, "It's 12. See, we don't measure it around, we measure it like this." She measures it again, across the top, and says, "It's 12 to me."

In the beginning of the year, we did some work in measurement. We measured each other's heights using paper strips and then I measured each child using a yardstick. Shelly and Joan have transferred this knowledge from these lessons to their own free play. They are very aware of each other's measurements and try to model for each other the "right" way to measure the bin.

Some Major Features of EM

Several often overlooked or misunderstood features of EM are important to highlight.

First, it is comprehensive. EM not only involves number, but also shape, space, measurement, magnitude and the like. Although researchers have tended to focus on number (for comprehensive reviews, see Baroody, Lai, & Mix, in press; Geary, 1994; Ginsburg, Cannon, Eisenband, & Pappas, 2005; Nunes & Bryant, 1996), children's interests are broader. For example, as is widely observed (Leeb-Lundberg, 1996) in block play, children often create *patterns* evident in constructions symmetrical in three dimensions and involving regular repetitions of shapes. Children also demonstrate competence in spatial relations. Thus, preschoolers can use external guides such as an informal X and Y axis to help specify location (Clements, Swaminathan, Hannibal, & Sarama, 1999). EM includes *measurement* too. Young children are vitally concerned with growing both bigger and older (Corsaro, 1985). Preschool students "…sometimes discussed eagerly, 'Who is the tallest?' with a keen sense of rivalry" (Isaacs, 1930, p. 41).

In short (a spatial, not numerical metaphor), children's EM is broad, including budding proficiency in number, shape, pattern, space and measurement, and no doubt other topics. It is certainly a mistake to limit our conception of EM to “numeracy.” Certainly, children’s play reflects the breadth of their mathematical interests.

Second, as we saw, children’s EM was often competent, as when Steven counted very high or Michelle noted a pattern. Contemporary research has stressed the young child’s competence in many aspects of EM (Gelman & Brown, 1986). For example, babies (Wynn, 1998) and children as young as 24 months (Sophian & Adams, 1987) have a basic understanding of adding and taking away. Preschoolers commonly use various strategies to calculate simple addition and subtraction problems (Carpenter, Moser, & Romberg, 1982). Thus, in trying to answer the question, “How much is three apples and two apples?” children may not only count on from the larger number (“three, and then four, five”) but also use such “derived facts” as “I know that two and two is four, and there is one more, so the answer is five” (Baroody & Dowker, 2003).

Research on children’s competence has made an enormous contribution. It has opened our eyes to the fact that young children are surprisingly proficient in at least some aspects of EM and suggests the possibility that young children can learn much more than we previously expected. Indeed, the research has so effectively introduced these insights that we now run the risk of exaggerating young children’s competence.

Third, although young children are indeed competent in many ways, their EM suffers at the same time from several weaknesses. Matthew struggles to figure out the meaning of “a lot.” Steven makes mistakes in counting. Derek cannot read 110. Researchers concur that competence and limits on competence coexist in young children’s minds. They understand principles underlying whole numbers (Gelman & Gallistel, 1986), but exhibit serious misunderstanding of rational numbers (Hartnett & Gelman, 1998). They can correctly locate clusters of model furniture items in a scale model of their classroom, but get confused when they must themselves position the items (Golbeck, Rand, & Soundy, 1986).

Also, despite strong critiques (e. g., Donaldson, 1978), we should not forget the substantial body of Piaget's research (e. g., Piaget, 1952a; Piaget & Inhelder, 1967) showing that children do indeed have clear cognitive limitations. The paradigmatic example is the preoperational child who cannot "conserve" numerical or other kinds of equivalence. Shown a line of 7 cups, each in a saucer, the preoperational child judges that the numbers of cups and saucers are the same. But when the cups are removed from the saucers and spread apart to form a line longer than the line of saucers, the preoperational child now believes that there are more cups than saucers. Even correctly counting the number in each line does not lead to recognition of the numerical equivalence. Thus, the preoperational child focuses on the *appearance* of the lines of cups and saucers, *centers* only on the dimension of length and ignores the spacing between elements, does not *reverse* thought to reason that because each cup could be returned to its corresponding saucer the numbers must be the same, and does not *understand* the significance of counting the two rows. In brief, Piaget's work shows that at least under some conditions (albeit perhaps more restricted than he originally proposed), young children's mathematical thinking is indeed limited.

Fourth, EM is sometimes very concrete or grounded in ordinary activities, as when children compare the heights of two block towers or try to grab the biggest cookie. But it also can be very abstract and in a real sense purely mathematical, as when children want to count to one hundred or what know what is the "largest number" (Gelman, 1980). Another way to say this is that young children do not necessarily require "manipulatives" to learn mathematics. They can learn from saying or hearing counting numbers or seeing visual patterns. As Piaget

(1970) pointed out, the child may learn from manipulating *ideas*, not necessarily objects: "…the most authentic research activity may take place in the spheres of reflection, of the most advanced abstraction…" (p.68).

Fifth, some aspects of EM are verbal, the most obvious being counting or knowing the names for the plain plane shapes, like "circle" or "square." But EM may sometimes take non-verbal form, as when Chris and Jeff build a block structure. Without speaking, they carefully attend to the lengths of the blocks, the positions where they place the blocks, the arrangement of the blocks (some have to be at right angles to others), and the geometrical nature of the blocks (cylinders are used for some purposes, rectangular prisms for others). Clearly they do not know the words for many aspects of their EM (e. g., "rectangular prism" and "cylinder"). The clearest example of non-verbal EM involves babies, who of course completely lack language, but can nevertheless determine that one set of dots is more numerous than another (Antell & Keating, 1983) and may be able to do a form of addition (Wynn, 1998).

How Common Is EM?

The examples of children's play show that EM is comprehensive, competent and at the same time limited, concrete and abstract, and both verbal and non-verbal. But the examples do not show how frequently mathematical activity occurs in children's everyday lives. One study attempted to determine the nature and frequency of young children's everyday mathematical activities and the extent to which they are associated with SES (Seo & Ginsburg, 2004). The investigators videotaped (for 15 minutes each) the everyday mathematical behavior of 90 individual 4- and 5-year-old children drawn about equally from lower-, middle-, and upper-SES families during free play in their daycare/preschool settings. Inductive methods were used to develop a coding system intended to capture the mathematical content of the children's behavior.

Three categories of mathematical activity occurred with some frequency. *Pattern and shape* (exploration of patterns and spatial forms) occurred during an average of about 21% of the 15 minutes; *magnitude* (statement of magnitude or comparison of two or more items to evaluate relative magnitude) during about 13 % of the minutes; and *enumeration* (numerical judgment or quantification) during about 12% of the minutes. No significant SES differences emerged in mathematical activity. I do not wish to exaggerate the extent to which mathematical activity occurs in free play: Note that several different categories of mathematical activity could occur during any given minute and that each of the activities could be of short duration. Nevertheless, it is fair to conclude that, regardless of SES, young children spontaneously and relatively frequently (albeit sometimes briefly) engage in forms of everyday mathematical activity ranging from counting to pattern.

Using Play In Early Childhood Mathematics Education

We have seen that very young children have an EM that permeates their play. Now the question is: what does all this mean for the goals and methods of early childhood mathematics education (ECME)? How can we use the knowledge gained from this research to improve ECME?

Background

Around the world, there is widespread interest in ECME. In the U.S., many states and other education agencies have introduced new literacy and mathematics programs for preschool children. Psychologists and educators have created research-based programs of early mathematics instruction (Casey, 2004; Greenes, Ginsburg, & Balfanz, 2004; Griffin, 2004; Serama & Clements, 2004; Sophian, 2004; Starkey, Klein, & Wakeley, 2004).

One major goal of these programs is to prepare children for school. The primary reason for the contemporary emphasis on this goal is that many education professionals, parents, and policy makers are concerned that American children's mathematics performance is weaker than it should be. East Asian children outperform their American counterparts in mathematics achievement, perhaps as early as kindergarten (Stevenson, Lee, & Stigler, 1986). Also, within the U.S., low-income and disadvantaged minority children show lower average levels of academic achievement than do their middle- and upper-income peers (Denton & West, 2002).

Clearly, American children in general, and low-income children in particular, should receive a better mathematics education than they do now. One part of a solution to the problem is quality mathematics instruction beginning in preschool. Research shows that a solid foundation in preschool education, including mathematics, can help to improve academic achievement for all children (Bowman, Donovan, & Burns, 2001). Of course, ECME cannot produce miracles; the mathematics instruction children receive once they arrive in school needs improvement too.

But what form should ECME take? Research on EM and on its role in children's play can help us answer this question.

Broadening the Goals of ECME

As noted, the main goal cited for ECME has been preparing young children for school in order to improve their later mathematics achievement. No doubt preparation for school is an important goal, especially for low-income children. But an exaggerated focus on the future can be self-defeating. It entails the danger of ignoring and even spoiling the present and thereby ultimately limiting what can be accomplished in the future. As Dewey (1938) put it:

> "What, then, is the true meaning of preparation in the educational scheme? In the first place, it means that a person, young or old, gets out of his present experience all that there is in it for him at the time in which he has it. When preparation is made the controlling end, then the potentialities of the present are sacrificed to a suppositious future. When this happens, the actual preparation for the future is missed or distorted" (p. 49).

We have seen that children's EM is exciting and vital. Young children develop mathematical strategies, grapple with important mathematical ideas, use mathematics in their play and play with mathematics. Young children often enjoy their mathematical work and play. Indeed, despite its immaturity, young children's mathematics bears some resemblance to research mathematicians' activity. Both young children and mathematicians ask and think about deep questions, invent solutions, apply mathematics to solve real problems, and play with mathematics. Clearly then, one of our goals should be to encourage and foster young children's *current* mathematical activities.

Indeed, if by contrast the exclusive goal is to prepare young children for the future, we run the risk of ignoring and even stifling children's current mathematical development. This can happen if we convert preschool into a miniature version of what passes for mathematics education in the higher grades. As Dewey (1976) put it, "The source of whatever is dead, mechanical, and formal in schools is found precisely in the subordination of the life and experience of the child to the curriculum" (p. 277). The mathematics of the schools is often a dreary chore, preserving little of the excitement and intellectual depth of young children's and research mathematicians' sometimes playful endeavors. Thus, if we drill preschoolers in number facts, we may increase their current and subsequent scores on tests that emphasize this topic (thus achieving high predictive "validity"—the validity of the trivial), but we may at the same time fail to foster their current more genuine mathematical interests and even instill at an earlier age than usual a virulent antipathy for the subject. In other words, a focus on preparation for school may allow us to achieve later success (narrowly defined) at the expense of real mathematics education.

The Content and Challenge of ECME

The research on EM suggests two simple lessons about the content that ECME should cover. One is that it should be broad, dealing not only with number and simple shape, but also with space, measurement, operations on numbers, and perhaps other topics as well. If children explore these topics on their own, there is good reason to include them in the curriculum. The second lesson is that the curriculum can be much challenging than it is now. Children like to count to high numbers, to read and write numerals, to explore symmetries in three dimensions. There is no need to limit so severely our and their expectations about what they can accomplish, especially when mastery of difficult problems can improve children's motivation to learn (Stipek, 1998).

Understand and Build on Children's EM

One of the major themes of early childhood education is "child-centered instruction." Following this approach, the teacher needs to take the child's perspective, understand the child's current intellectual activities, and build on them to foster the child's learning, whether of mathematics or any other topic. "…[T]eachers need to find out what young children already understand and help them begin to understand these things mathematically" (National Association for the Education of Young Children and National Council of Teachers of Mathematics, 2002, p. 6). Play is an especially promising setting for child-centered teaching.

“Play does not guarantee mathematical development, but it offers rich possibilities. Significant benefits are more likely when teachers follow up by engaging children in reflecting on and representing the mathematical ideas that have emerged in their play” (ibid, p. 10).

A popular early childhood program, Creative Curriculum, follows this approach (Dodge, Colker, & Heroman, 2002). Although this is an admirable strategy, it is difficult to implement. It requires first that teachers recognize children’s EM in real time during play and second that they then seize upon the teachable moment to foster children’s learning. Clearly, early childhood teachers who by and large have had little acquaintance with or training in ECME require a good deal of help to make child-centered teaching a practical reality.

Introduce a Playful and Organized Mathematics Curriculum

We have seen that a child-centered approach involves recognizing and building upon the EM in children’s play and other activities. But this kind of child-centered approach is not sufficient. The teacher must do more than seize upon the teachable moment that arises spontaneously. “In high-quality mathematics education for 3- to 6-year-old children, teachers and other key professionals should… actively introduce mathematical concepts, methods, and language through a range of appropriate experiences and teaching strategies” (National Association for the Education of Young Children and National Council of Teachers of Mathematics, 2002, p. 4).

One way to do this is through the project approach (Edwards, Gandini, & Forman, 1993; Katz & Chard, 1989) in which teachers and children engage in large-scale activities like making applesauce, and then exploiting and elaborating on the mathematics and science that arise in the course of the activity. The strength of the project method is that it situates the learning of mathematics in a highly motivating investigation. But the weakness of the method is that alone, it does not constitute a coherent curriculum (Ginsburg & Golbeck, 2004). Projects can be exciting but do not structure the emerging ideas in a systematic way.

Therefore, in addition to building upon children's everyday mathematics and introducing conceptually rich projects, teachers should use a curriculum which "…is more than a collection of activities; it must be coherent, focused on important mathematics, and well articulated across the grades" (National Association for the Education of Young Children and National Council of Teachers of Mathematics, 2002, p. 2).

The problem then is how to teach a mathematics curriculum in a way that is appropriate for young children and in tune with their EM. What does the research on EM tell us about how to do this? Not a great deal, but it does suggest some guidelines. One is that the curriculum should be playful, in order to preserve the kind of natural enthusiasm that characterizes children's EM. The curriculum should cover a wide range of mathematics and need not be limited to the concrete. As we have seen, EM may involve abstract ideas. But whether concrete or abstract, the curriculum should be playful.

Big Math for Little Kids (BMLK) (Greenes et al., 2004), a curriculum designed for 4- and 5-year-olds, offers a pertinent example. BMLK offers a planned sequence of activities covering a large range of mathematical topics and is intended for use each day of the school year.

Consider a counting activity that is central to the BMLK approach to number. The activity derived from several observations. One is that children like to say the counting numbers, and in fact are often interested in counting as high as possible. Recall Steven's attempt to count 100 objects. Given children's interest in counting, we thought that we would foster it, and developed an activity, Counting with Pizzazz, designed to teach children, over the course of the year, to count to 100. Why 100? We ask them to count this high because young children see100 as a big number, and they are very proud to be able to reach it. At the Pre-K level, Counting with Pizzazz is done almost every day during the year, often at circle time. It takes only a few minutes, and as we shall see, it is a good physical activity for children (and teacher too).

We begin the activity by practicing the number words "one" through "ten." In English, and in virtually all other languages, these numbers must be memorized. There is no sense to the first ten numbers (and also to "eleven" and "twelve"). After that point, English counting becomes more regular and operates according to system of base ten rules. We usually say the "decade" word, like "twenty" or "fifty," and then add on to it the unit words "one, two…nine." The numbers from 20 to 99 are fairly regular. In English, the numbers from 11 to 20 are very odd. In fact, most of them are "backwards." "Thirteen" should be "teen-three," just like "twenty-three" and "forty-three." In brief, the numbers from 1 to 12 or so must be memorized; the numbers from 13 to 19 are backwards; and the numbers from 20 to 99 are governed by base ten rules. From an educational point of view it is ironic that although the easiest numbers to learn are those 20 and above, we first teach children the number words that make no sense and then the ones that violate the important base-ten rules.

BMLK helps children learn to count by engaging in various physical activities as they say the numbers. For example, they can jump from 1 to 9 or raise the left hand for each number, then hop from 10 to 19, raise their arms from 20 to 29 and so on. Each day, the activity can be varied; sometimes the children choose them. Each class does the activity differently and sets its own time schedule. One class may spend a month on the numbers from 1 to 9, and another class may spend two months going from 11 to 19. Different classes may make different faces and sounds to mark the decades (the tearful twenties and ferocious forties).

A second observation that shaped our approach to teaching counting is that children often enjoy playing with written numerals. We observed one 3-year-old who spontaneously chose to put in order a collection of number cards from 1 to 30. He did this day after day, and eventually achieved a good amount of success. Given this observation, and given our desire to help children learn the pattern underlying the system of counting numbers, we chose to present written numerals as children count. When they learn a new set of numbers, whether from 1 to 9 or 50 to 59, the teacher helps them construct a new portion of the number chart, with each number on a separate card. Then, as the children count, the teacher points to each number in turn, saying nothing else. After the counting activity is completed, the teacher makes the number chart available to the children during their free play. After a year of these kinds of activities, the children seem to learn both to count and to read most of the numerals to 100.

Is this play? On the one hand, the teacher directs the counting activity and the curriculum developers decided that the reading of numerals should be linked to saying the counting numbers. Clearly, the counting activities are not primarily student generated. At the same time, the material is presented in a playful manner, and the children can play with what the teacher has taught. Recall the example reported by Luzaria Dunatov, whose students enjoyed the game of testing each other on reading numbers as they played teacher and student.

Policy Implications

•*Support development of new and innovative curricula and make them available*

At the present time, few mathematics curricula for young children are available. Work in this area is just beginning. We should invest in developing new and innovative curricula. We should also make them available to the preschools and child care centers that serve an increasingly large proportion of the preschool population. High quality preschool education requires funding at least at the level of good elementary education.

•Strengthen teacher professional development

Preschool teachers need extensive professional development to learn to implement early childhood mathematics education effectively. Professional development should promote an understanding of children's EM, as well as mathematics itself and pedagogy (Ginsburg, Kaplan et al., 2005). Students of education in colleges and universities also need to acquire this knowledge and methods for helping them to do so are being developed (Ginsburg, Jang, Preston, Appel, & VanEsselstyn, 2004).

•Create new forms of evaluation and assessment

Child-centered teaching and curriculum require deep understanding of children's EM and their learning of mathematics in an organized curriculum. Teachers need to learn effective methods of observation and clinical interview (Bowman et al., 2001). These methods are more valuable than standard tests for the purpose of improving everyday instruction. But some form of appropriate standard testing is required to evaluate the success of curricula. At present, few appropriate tests are available. We need to support their development (Hirsh-Pasek, Kochanoff, Newcombe, & de Villiers, 2005).

•Conduct teaching experiments in context

It is a truism to say that more research is needed. But it is. In particular, we need research that focuses not so much on what children know, but on what they could know under stimulating conditions. Good teaching experiments (e.g., Zur & Gelman, 2004; Zvonkin, 1992) are rare. We need more of them.

Conclusion

Many otherwise intelligent people suffer from fear and loathing of mathematics. One might even say that these feelings have been a cultural imperative in the U.S. Perhaps this is one reason why the idea of teaching mathematics to preschoolers arouses antipathy in some quarters. Indeed, many teachers seem to believe that early childhood mathematics education is an unnecessary, unpleasant and developmentally inappropriate imposition on young children. But we have seen that this need not be the case. Mathematics is embedded in children's play, just as it is in many aspects of their lives; children enjoy playing with everyday mathematics; and children even spontaneously play with the mathematics taught in school. Mathematics education for young children need not be dreadful. Early mathematics education need not focus only on preparation for future ordeals. Teaching mathematics to young children can be developmentally appropriate and enjoyable for child and teacher alike when it is challenging and playful and produces real learning.

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