Multidimensional Normal Distribution

**Definition:** The $p$-dimensional random vector $\mathbf{Z}$ called *standard normal* if the coordinates of $\mathbf{Z}$ are independent standard normal one-dimensional random variables, i.e. $\mathbf{Z} = (Z_1, \ldots, Z_p)$, $Z_i \sim \mathcal{N}(0,1)$, $i = 1, \ldots, p$.

**Theorem 1.** If $\mathbf{Z} \in \mathbb{R}^p$ is a random vector with standard normal distribution then the following statements are equivalent

(i) The density of $\mathbf{Z}$ is given by

$$f(Z_1, \ldots, Z_p) = (2\pi)^{-\frac{p}{2}} \exp\{-\frac{1}{2} \mathbf{Z}' \mathbf{Z}\}.$$

(ii) Let $R_p(\mathbf{u}) = \mathbb{E}(\exp\{\mathbf{u}' \mathbf{Z}\})$ be the moment generating function of $\mathbf{Z}$, then

$$R_p(\mathbf{u}) = \exp\{\frac{1}{2} \mathbf{u}' \mathbf{u}\}.$$

(iii) For arbitrary $\mathbf{c} \in \mathbb{R}^p$ where $\mathbf{c}$ is a constant vector, $\mathbf{c}' \mathbf{Z}$ is a one dimensional random variable with normal distribution; $\mathbb{E}(\mathbf{c}' \mathbf{Z}) = 0$, $\text{Var}(\mathbf{c}' \mathbf{Z}) = \mathbf{c}' \mathbf{c}$, i.e. $\mathbf{c}' \mathbf{Z} \sim \mathcal{N}(0, \mathbf{c}' \mathbf{c})$.

(iv) Let $\mathbf{U}$ be a $p \times p$ orthogonal matrix, then the distribution of $\mathbf{Z}$ and $\mathbf{UZ}$ are the same and $\mathbf{Z}' \mathbf{Z}$ has a $\chi^2$ distribution with $p$ degree of freedom.

**Definition:** The random vector $\mathbf{X} \in \mathbb{R}^p$ has *normal distribution* if it can be expressed as

$$\mathbf{X} = \mathbf{A} \mathbf{Z} + \mathbf{\mu},$$

where $\mathbf{A}$ is a $p \times \ell$ constant matrix, $\mathbf{Z} \in \mathbb{R}^{\ell}$, $\ell \leq p$ is a random vector with standard normal distribution and $\mathbf{\mu} \in \mathbb{R}^p$ is a constant vector.

**Remarks:**

1. $\mathbb{E}\mathbf{X} = \mathbf{\mu}$.

2. $\Sigma_\mathbf{X} = \mathbb{E}((\mathbf{X} - \mathbb{E}(\mathbf{X}))(\mathbf{X} - \mathbb{E}(\mathbf{X}))') = \mathbb{E}((\mathbf{A} \mathbf{Z})(\mathbf{A} \mathbf{Z})') = \mathbb{E}(\mathbf{A} \mathbf{Z} \mathbf{Z}' \mathbf{A}') = \mathbf{A} \mathbf{I}_\ell \mathbf{A}' = \mathbf{A} \mathbf{A}'$, where $\mathbf{I}_\ell$ denotes the $\ell \times \ell$ matrix with 1-s in the main diagonal and 0-s elsewhere.
Theorem 2. The multidimensional normal distribution is uniquely defined by its expected value vector and its covariance matrix.

Proof. There is a one to one correspondence between moment generating function and distribution.

\[ E(\exp(u'X)) = E(\exp(u'AZ + u'\mu)) = e^{u'\mu}e^{\frac{1}{2}u'AA'u}. \]

The last equality is true since by definition \( E(\exp(u'AZ)) \) is the moment generating function of \( Z \) at \( A'u \).

Notation: \( X \in \mathbb{R}^p \) random variable with normal distribution, with expected value \( E X = \mu \) and covariance matrix \( \Sigma_X \) is denoted as

\[ X \sim \mathcal{N}_p(\mu, \Sigma_X). \]

Remark: Moment generating function of \( X \):

\[ R_p(u) = E(\exp\{u'X\}) = e^{u'\mu}e^{\frac{1}{2}u'Su}. \]

Definition: Let \( X \sim \mathcal{N}_p(\mu, \Sigma_X) \), then

\[ X = AZ + \mu \]

is the canonical form of \( X \) if \( A \) has the following properties:

(i) the column vectors of \( A \) are orthogonal with positive norms,

(ii) the norms of column vectors of \( A \) are non-increasing.

Theorem 3. Any \( X \in \mathbb{R}^p \) normally distributed random variable has canonical form.

Proof. Let \( X \sim \mathcal{N}_p(\mu, \Sigma_X) \), then by definition there exists a \( (p \times \ell) \) matrix \( A \) such that \( X = AZ + \mu \), where \( z \in \mathbb{R}^\ell \) standard normal random variable. Consider the singular value decomposition of \( A \). \( A = VDU' \) where \( V \) and \( U \) are \( p \times p \) and \( \ell \times \ell \) orthogonal matrices and \( D \) is a \( p \times \ell \) diagonal matrix where the values \( s_1 \geq s_2 \geq \ldots \geq 0 \) non-increasing in the diagonal. Thus

\[ X = VD(U'Z) + \mu. \]

Using Theorem 1 (iv), \( U'Z \) has the same distribution as \( Z \). It means that

\[ X = (VD)Z + \mu = AZ + \mu, \]

is a canonical representation of \( X \).

Theorem 4. Let \( X \sim \mathcal{N}_p(\mu_X, \Sigma_X) \) random variable. Then any linear transformation of \( X \) has normal distribution.

Proof. Let \( Y = BX + \mu_Y \). Then

\[ Y = B(AZ + \mu_X) + \mu_Y = (BA)Z + (B\mu_X + \mu_Y). \]