LIPS: LiDAR-Inertial 3D Plane SLAM

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1 Graph Based Optimization

1.1 IMU State Definition

Each of our poses is defined by a IMU state which is parameterized by the following 16×1 vector:

$$\mathbf{x}_I = \begin{bmatrix} {}^I_G\bar{q}^\top & \mathbf{b}_g^\top & {}^G\mathbf{v}_I^\top & \mathbf{b}_a^\top & {}^G\mathbf{p}_I^\top \end{bmatrix}^\top \tag{1}$$

where the quaternion ${}^I_G\bar{q}$ represents the rotation from global to the IMU, the velocity is of the IMU seen from the global frame, position is of the IMU seen from the global frame, and $\mathbf{b}_g$ and $\mathbf{b}_a$ are the gyroscope and accelerometer biases respectively.

To perform state estimation, we need the define our error state that will correct the current state estimate during each update step. Note that special care needs to be taken to use a multiplicative error model for the quaternion. We can define the minimal representation error state as the following:

$$\widetilde{\mathbf{x}} = \begin{bmatrix} {}^I_G\boldsymbol{\delta\theta}^\top & \widetilde{\mathbf{b}}_g^\top & {}^G\widetilde{\mathbf{v}}_I^\top & \widetilde{\mathbf{b}}_a^\top & {}^G\widetilde{\mathbf{p}}_I^\top \end{bmatrix}^\top \tag{2}$$

where the quaternion error is defined by the 3×1 error vector defined as the following [1]:

$$\delta\bar{q} = \bar{q} \otimes \hat{\bar{q}}^{-1} \simeq \begin{bmatrix} \frac{1}{2}\boldsymbol{\delta\theta}^\top & 1 \end{bmatrix}^\top \tag{3}$$

1.2 Plane State Definition

Each plane is a single 3×1 vector representing the *closest* 3D plane point to the current frame. This "closest point" plane representation can be found in detail in in Section 2. We can define the following state and error state for a given plane:

$$\mathbf{x}_\Pi = {}^G\boldsymbol{\Pi} \tag{4}$$

$$\tilde{\mathbf{x}}_\Pi = \mathbf{x}_\Pi - \hat{\mathbf{x}}_\Pi \tag{5}$$

$$= {}^G\boldsymbol{\Pi} - {}^G\hat{\boldsymbol{\Pi}} \tag{6}$$

where the left superscript denotes the frame of reference the plane is seen from.

2 Closest Point Plane Representation

2.1 Representation Formulation

We choose the following plane representation:

$$ {}^G\boldsymbol{\Pi} = \begin{bmatrix} {}^G\Pi(x) \\ {}^G\Pi(y) \\ {}^G\Pi(z) \end{bmatrix}_{3\times 1} \tag{7}$$

where ${}^G\boldsymbol{\Pi}_i$ is the *closest* 3D plane point to the current frame of reference. This representation can be related to the "conventional" Hesse representation through the following mappings:

$$\text{fromND}\left(\begin{bmatrix} {}^G\mathbf{n}^\top & {}^Gd \end{bmatrix}^\top \right): \ {}^G\boldsymbol{\Pi} = {}^G\mathbf{n}\ {}^Gd \tag{8}$$

$$\text{toND}\left({}^G\boldsymbol{\Pi}\ \right): \ \begin{bmatrix} {}^G\mathbf{n} \\ {}^Gd \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{{}^G\Pi(x)^2 + {}^G\Pi(y)^2 + {}^G\Pi(z)^2}}\ {}^G\boldsymbol{\Pi} \\ \sqrt{{}^G\Pi(x)^2 + {}^G\Pi(y)^2 + {}^G\Pi(z)^2} \end{bmatrix} \tag{9}$$
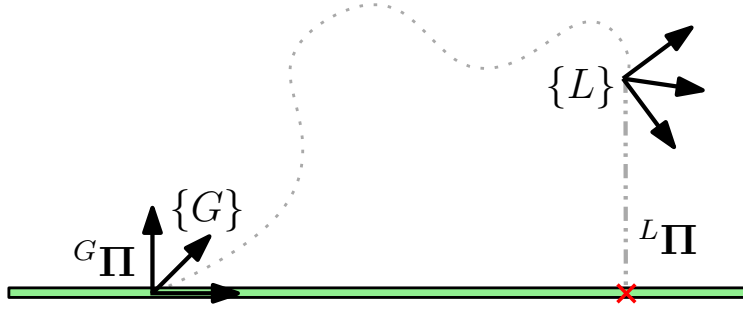
2.2 Singularity Discussion

Figure 1: An example situation where the local plane parameter $^L\mathbf{\Pi}$ is well defined while the global plane representation $^G\mathbf{\Pi}$ is ill-defined as the found plane intercepts the global coordinate frame. We show that this measurement, $^L\mathbf{\Pi}$, is not ill-defined if it remains in the *local* sensor reference frame.

The above representation has a singularity when the value of ^Gd approaches zero. Any plane $^G\mathbf{\Pi}$ that intersects our frame of reference $\{G\}$ will be represented as the same zero vector regardless of the plane's orientation since the closest point on that plane is at zero. This singularity can cause an issue if the plane is represented in a "global" frame of reference. An example failure case is if the system detects a plane in its local frame of reference that when transformed into the global frame intersects its origin (see Figure 1).

This issue can be avoided by ensuring that the planes are represented in their first seen frame of reference, in this case the local frame $\{L\}$. We argue that this singularity is well suited in the case of plane estimation using range based sensors (e.g., LiDAR and RGBD cameras) since planes extracted from these sensors should not be ill-defined when represented the *local* sensor frame. Zhang et al. [2] also noted that planes that are close to intersecting the sensor frame should be consider "unreliable" if found and discarded.

3 Point Cloud to Plane Compression

From a range based sensor, we have a set of 3D points, $\{\mathbf{PT}\}$, that belong to a plane in the state. To leverage this possible large point cloud, we would like to first compress this set of 3D points into a local measurement of the plane parameters. This local plane measurement can then be inserted into the graph as a measurement of the estimated plane. Given a set of unorder points, we can calculate the local parameters as follows:

$$^L\mathbf{\Pi}^* = \operatorname*{argmin}_{^L\mathbf{\Pi}} \sum_{i=1}^{n} \left\| \ r_i(^L\mathbf{\Pi}) \ \right\|^2_{W_i(^L\mathbf{\Pi})} \tag{10}$$

$$= \operatorname*{argmin}_{^L\mathbf{\Pi}} \sum_{i=1}^{n} \left\| ^L\mathbf{n}^{\top L}\mathbf{p}_{mi} - {^L}d \right\|^2_{W_i(^L\mathbf{\Pi})} \tag{11}$$

$$= \operatorname*{argmin}_{^L\mathbf{\Pi}} \sum_{i=1}^{n} \left\| \frac{^L\mathbf{\Pi}^{\top}}{\sqrt{^L\mathbf{\Pi}^{\top L}\mathbf{\Pi}}} {^L}\mathbf{p}_{mi} - \sqrt{^L\mathbf{\Pi}^{\top L}\mathbf{\Pi}} \right\|^2_{W_i(^L\mathbf{\Pi})} \tag{12}$$

$$= \operatorname*{argmin}_{^L\mathbf{\Pi}} \sum_{i=1}^{n} \left\| \frac{^L\mathbf{\Pi}^{\top}}{\|^L\widehat{\mathbf{\Pi}}\|} {^L}\mathbf{p}_{mi} - \|^L\widehat{\mathbf{\Pi}}\| \right\|^2_{W_i(^L\mathbf{\Pi})} \tag{13}$$

where we define $r_i(\cdot)$ as the residual of a measurement i, $W_i({}^L\mathbf{\Pi})$ as the information matrix of the noise that corrupts the i'th measurement (see Appendix A), ${}^L\mathbf{\Pi}$ as the point on the plane that that is the shortest distance to that plane in the local frame, and ${}^L\mathbf{p}_{mi}$, $i \in \{\mathbf{PT}\}$ is a 3D point on the plane seen from the local frame. This cost function minimizes the scalar difference between the point ${}^L\mathbf{p}_{mi}$ projected onto the plane normal, ${}^L\mathbf{n} = \frac{{}^L\mathbf{\Pi}}{\|{}^L\widehat{\mathbf{\Pi}}\|}$, and the shortest distance to the plane, ${}^L d = \|{}^L\widehat{\mathbf{\Pi}}\|$.

We minimize the cost function using the Gauss-Newton method of iterative linearization of the residual about the current best estimate. Formally, we solve for the correction vector, ${}^L\widetilde{\mathbf{\Pi}}$, to our linearization point ${}^L\hat{\mathbf{\Pi}}$:

$$
{}^L\widetilde{\mathbf{\Pi}}^* = \operatorname*{argmin}_{{}^L\widetilde{\mathbf{\Pi}}} \sum_{i=1}^{n} \left|\left| \, r_i({}^L\hat{\mathbf{\Pi}} \boxplus {}^L\widetilde{\mathbf{\Pi}}) \, \right|\right|^2_{W_i({}^L\mathbf{\Pi})} \tag{14}
$$

$$
\simeq \operatorname*{argmin}_{{}^L\widetilde{\mathbf{\Pi}}} \sum_{i=1}^{n} \left|\left| \, r_i({}^L\hat{\mathbf{\Pi}}) + \mathbf{J}_i \, {}^L\widetilde{\mathbf{\Pi}} \, \right|\right|^2_{W_i({}^L\mathbf{\Pi})} \tag{15}
$$

where $\mathbf{J}_i$ is the Jacobian of the residual in respect to the local plane error state (see Appendix C):

$$
\mathbf{J}_i = \frac{\partial \, r_i}{\partial {}^L\widetilde{\mathbf{\Pi}}} \bigg|_{{}^L\hat{\mathbf{\Pi}}} \tag{16}
$$

$$
= \frac{{}^L\mathbf{p}_{mi}^{\top}}{\|{}^L\widehat{\mathbf{\Pi}}\|} - \left({}^L\mathbf{p}_{mi}^{\top} {}^L\mathbf{\Pi}\right) \frac{{}^L\widehat{\mathbf{\Pi}}^{\top}}{\|{}^L\widehat{\mathbf{\Pi}}\|^3} - \frac{{}^L\widehat{\mathbf{\Pi}}^{\top}}{\|{}^L\widehat{\mathbf{\Pi}}\|} \tag{17}
$$

From Appendix B, we know that by taking the gradient of the linearized cost function, equation (15), and finding where it is zero gives us the following correction vector:

$$
{}^L\widetilde{\mathbf{\Pi}} = -\left(\sum_{i=1}^{n} \mathbf{J}_i^{\top} \, W_i \, \mathbf{J}_i \right)^{-1} \left(\sum_{i=1}^{n} \mathbf{J}_i^{\top} W_i \, r_i({}^L\hat{\mathbf{\Pi}}) \right) \tag{18}
$$

Using equation (18), we can update our current estimate. For each iteration we compute the weights, Jacobians, and update till we converge to a solution. We update our current estimate using the following linear update function:

$$
{}^L\widehat{\mathbf{\Pi}}^{+} = {}^L\widehat{\mathbf{\Pi}}^{-} + {}^L\widetilde{\mathbf{\Pi}} \tag{19}
$$

In summary, using the set of 3D points, $\{\mathbf{PT}\}$, we have calculate the local plane parameters and its covariance matrix and will define them as the following:

$$
{}^L\mathbf{\Pi} = {}^L\widehat{\mathbf{\Pi}} + \mathbf{n}_{\Pi} \tag{20}
$$

$$
\text{where } \mathbf{n}_{\Pi} \sim \mathcal{N}(0, \mathbf{P}_{\Pi}) \tag{21}
$$

$$
\text{where } \mathbf{P}_{\Pi} = \left(\sum_{i=1}^{n} \mathbf{J}_i^{\top} \, W_i \, \mathbf{J}_i \right)^{-1} \tag{22}
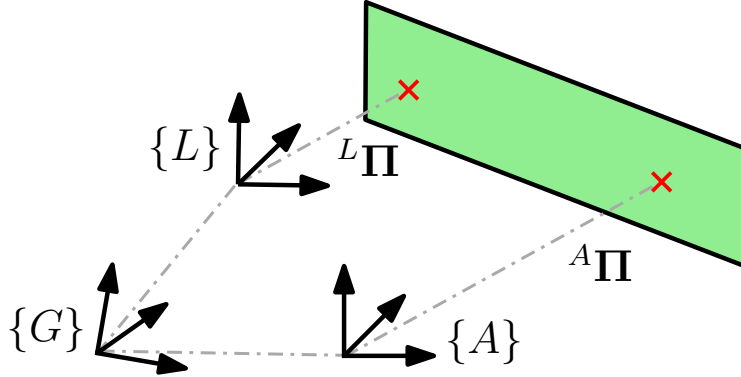$$

4   Anchor Plane Graph Factor

Figure 2: Pictorial view of a plane seen in the local $\{L\}$ frame which can be transformed into its anchor frame $\{A\}$ in which we know we have a well-defined plane representation that can be used for optimization.

We now have the local plane parameters seen in the local frame of reference at time k and would like to update the value of the global plane parameters in the state. It is important to note that this is not a simple coordinate transform of a 3D point. Each frame of reference has its own *closest* point on the plane, thus, the plane representation in the local frame $\{L\}$ will not be the same as $\{G\}$ or $\{A\}$. We can define the following equation that will map the plane seen in the local frame into the of the anchor pose. We define the coordinate transform between the anchor frame, $\{A\}$, and local frame of reference, $\{L\}$, as:

$$\begin{bmatrix} {}^L\mathbf{n} \\ {}^L d \end{bmatrix} = \begin{bmatrix} {}^L_A\mathbf{R} & 0 \\ -{}^A\mathbf{p}_L^\top & 1 \end{bmatrix} \begin{bmatrix} {}^A\mathbf{n} \\ {}^A d \end{bmatrix} \tag{23}$$

where the relative transform is defined by:

$$ {}^L_A\mathbf{R} = {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^\top \tag{24}$$

$$ {}^A\mathbf{p}_L = {}^A_G\mathbf{R}\left({}^G\mathbf{p}_L - {}^G\mathbf{p}_A\right) \tag{25}$$

Using equation (8), we can defined the combined measurement equation as the following:

$$ {}^L\mathbf{\Pi} = h(\mathbf{x}_A, \mathbf{x}_L, \mathrm{toND}(\mathbf{x}_\Pi)) \tag{26}$$

$$ = \left({}^L_A\mathbf{R}^A\mathbf{n}\right)\left(-{}^A\mathbf{p}_L^{\top\,A}\mathbf{n} + {}^A d\right) \tag{27}$$

$$ = {}^L_A\mathbf{R}^A\mathbf{n}^A d - {}^L_A\mathbf{R}^A\mathbf{n}^A\mathbf{p}_L^{\top\,A}\mathbf{n} \tag{28}$$

$$ = {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^{\top\,A}\mathbf{n}^A d - {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^{\top\,A}\mathbf{n}\left({}^A_G\mathbf{R}\left({}^G\mathbf{p}_L - {}^G\mathbf{p}_A\right)\right)^{\top\,A}\mathbf{n} \tag{29}$$

$$ = {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^{\top\,A}\mathbf{n}^A d - {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^{\top\,A}\mathbf{n}\left({}^G\mathbf{p}_L - {}^G\mathbf{p}_A\right)^{\top\,A}_G\mathbf{R}^{\top\,A}\mathbf{n} \tag{30}$$

$$ = {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^{\top\,A}\mathbf{n}^A d - {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^{\top\,A}\mathbf{n}^G\mathbf{p}_L^\top\, {}^A_G\mathbf{R}^{\top\,A}\mathbf{n} + {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^{\top\,A}\mathbf{n}^G\mathbf{p}_A^\top\, {}^A_G\mathbf{R}^{\top\,A}\mathbf{n} \tag{31}$$

$$ = {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^\top\left({}^A\mathbf{n}^A d - {}^A\mathbf{n}^G\mathbf{p}_L^\top\, {}^A_G\mathbf{R}^{\top\,A}\mathbf{n} + {}^A\mathbf{n}^G\mathbf{p}_A^\top\, {}^A_G\mathbf{R}^{\top\,A}\mathbf{n}\right) \tag{32}$$

The above equation maps the anchor plane representation, $^A\boldsymbol{\Pi}$, into the local, $^L\boldsymbol{\Pi}$, where $^A\mathbf{n}$ and ^Ad can be found using equation (9). From here we perform a Taylor series expansion to linearize the measurement function:

$$\mathbf{z}_m = h(\mathbf{x}_A, \mathbf{x}_L, \text{toND}(\mathbf{x}_\Pi)) + \mathbf{n}_\Pi \tag{33}$$

$$= h(\hat{\mathbf{x}}_A \boxplus \tilde{\mathbf{x}}_L, \hat{\mathbf{x}}_L \boxplus \tilde{\mathbf{x}}_A, \text{toND}(\hat{\mathbf{x}}_\Pi + \tilde{\mathbf{x}}_\Pi)) + \mathbf{n}_\Pi \tag{34}$$

Our estimated error in the error states is zero, giving us the following Taylor series:

$$\simeq h(\hat{\mathbf{x}}_A \boxplus \mathbf{0}, \hat{\mathbf{x}}_L \boxplus \mathbf{0}, \text{toND}(\hat{\mathbf{x}}_\Pi + \mathbf{0})) + \frac{\partial h}{\partial \tilde{\mathbf{x}}_A}\Big|_{\hat{\mathbf{x}}_A, \hat{\mathbf{x}}_L, \hat{\mathbf{x}}_\Pi} (\tilde{\mathbf{x}}_A - \mathbf{0}) + \frac{\partial h}{\partial \tilde{\mathbf{x}}_L}\Big|_{\hat{\mathbf{x}}_A, \hat{\mathbf{x}}_L, \hat{\mathbf{x}}_\Pi} (\tilde{\mathbf{x}}_L - \mathbf{0})$$

$$+ \frac{\partial h}{\partial \text{toND}(\cdot)}\Big|_{\hat{\mathbf{x}}_A, \hat{\mathbf{x}}_L, \hat{\mathbf{x}}_\Pi} \frac{\partial \text{toND}(\cdot)}{\partial \tilde{\mathbf{x}}_\Pi}\Big|_{\hat{\mathbf{x}}_A, \hat{\mathbf{x}}_L, \hat{\mathbf{x}}_\Pi} (\tilde{\mathbf{x}}_\Pi - \mathbf{0}) + \mathbf{n}_\Pi \tag{35}$$

$$\simeq h(\hat{\mathbf{x}}_A, \hat{\mathbf{x}}_L, \text{toND}(\hat{\mathbf{x}}_\Pi)) + \frac{\partial h}{\partial \tilde{\mathbf{x}}_A}\Big|_{\hat{\mathbf{x}}_A, \hat{\mathbf{x}}_L, \hat{\mathbf{x}}_\Pi} (\tilde{\mathbf{x}}_A) + \frac{\partial h}{\partial \tilde{\mathbf{x}}_L}\Big|_{\hat{\mathbf{x}}_A, \hat{\mathbf{x}}_L, \hat{\mathbf{x}}_\Pi} (\tilde{\mathbf{x}}_L)$$

$$+ \frac{\partial h}{\partial \text{toND}(\cdot)}\Big|_{\hat{\mathbf{x}}_A, \hat{\mathbf{x}}_L, \hat{\mathbf{x}}_\Pi} \frac{\partial \text{toND}(\cdot)}{\partial \tilde{\mathbf{x}}_\Pi}\Big|_{\hat{\mathbf{x}}_A, \hat{\mathbf{x}}_L, \hat{\mathbf{x}}_\Pi} (\tilde{\mathbf{x}}_\Pi) + \mathbf{n}_\Pi \tag{36}$$

$$\simeq h(\hat{\mathbf{x}}_A, \hat{\mathbf{x}}_L, \text{toND}(\hat{\mathbf{x}}_\Pi)) + \mathbf{H}_A \tilde{\mathbf{x}}_A + \mathbf{H}_L \tilde{\mathbf{x}}_L + \mathbf{H}_{nd}\mathbf{H}_\Pi \tilde{\mathbf{x}}_\Pi + \mathbf{n}_\Pi \tag{37}$$

where $\text{toND}(\cdot)$ is defined by Equation (9). Taking the expected value we have the following expected measurement:

$$\mathbb{E}\left[\mathbf{z}_m\right] = \mathbb{E}\left[h(\hat{\mathbf{x}}_A, \hat{\mathbf{x}}_L, \text{toND}(\hat{\mathbf{x}}_\Pi)) + \mathbf{H}_A \tilde{\mathbf{x}}_A + \mathbf{H}_L \tilde{\mathbf{x}}_L + \mathbf{H}_{nd}\mathbf{H}_\Pi \tilde{\mathbf{x}}_\Pi + \mathbf{n}_\Pi\right] \tag{38}$$

$$\hat{\mathbf{z}}_m = h(\hat{\mathbf{x}}_A, \hat{\mathbf{x}}_L, \text{toND}(\hat{\mathbf{x}}_\Pi)) \tag{39}$$

The Jacobians are defined by the following (see Appendix D for derivations):

$$\mathbf{H}_A = \begin{bmatrix} \mathbf{H}_{A1} & \mathbf{H}_{A2} \end{bmatrix}_{3\times 6} \tag{40}$$

$$\mathbf{H}_L = \begin{bmatrix} \mathbf{H}_{L1} & \mathbf{H}_{L2} \end{bmatrix}_{3\times 6} \tag{41}$$

$$\mathbf{H}_{nd} = \begin{bmatrix} \mathbf{H}_n & \mathbf{H}_d \end{bmatrix}_{3\times 4} \tag{42}$$

$$\mathbf{H}_\Pi = \begin{bmatrix} \mathbf{H}_{\Pi 1} \\ \mathbf{H}_{\Pi 2} \end{bmatrix}_{4\times 3} \tag{43}$$

$$\mathbf{H}_{A1} = {}_G^L\mathbf{R}\,{}_G^A\hat{\mathbf{R}}^\top\left({}^A\mathbf{n}\,{}^G\mathbf{p}_L^\top\,{}_G^A\hat{\mathbf{R}}^\top\lfloor{}^A\mathbf{n}\,\times\rfloor - {}^A\mathbf{n}\,{}^G\mathbf{p}_A^\top\,{}_G^A\hat{\mathbf{R}}^\top\lfloor{}^A\mathbf{n}\,\times\rfloor\right.$$
$$\left. - \lfloor{}^A\mathbf{n}^Ad\,\times\rfloor + \lfloor{}^A\mathbf{n}\,{}^G\mathbf{p}_L^\top\,{}_G^A\hat{\mathbf{R}}^{\top A}\mathbf{n}\,\times\rfloor - \lfloor{}^A\mathbf{n}\,{}^G\mathbf{p}_A^\top\,{}_G^A\hat{\mathbf{R}}^{\top A}\mathbf{n}\,\times\rfloor\right) \quad (44)$$

$$\mathbf{H}_{A2} = {}_G^L\mathbf{R}\,{}_G^A\mathbf{R}^{\top A}\mathbf{n}\,{}^A\mathbf{n}^\top{}_G^A\mathbf{R} \quad (45)$$

$$\mathbf{H}_{L1} = \left\lfloor {}_G^L\hat{\mathbf{R}}\,\,{}_G^A\mathbf{R}^\top\left({}^A\mathbf{n}^Ad - {}^A\mathbf{n}\,{}^G\mathbf{p}_L^\top\,{}_G^A\mathbf{R}^{\top A}\mathbf{n} + {}^A\mathbf{n}\,{}^G\mathbf{p}_A^\top\,{}_G^A\mathbf{R}^{\top A}\mathbf{n}\right)\times\right\rfloor \quad (46)$$

$$\mathbf{H}_{L2} = -{}_G^L\mathbf{R}\,{}_G^A\mathbf{R}^{\top A}\mathbf{n}\,{}^A\mathbf{n}^\top{}_G^A\mathbf{R} \quad (47)$$

$$\mathbf{H}_n = {}_G^L\mathbf{R}\,{}_G^A\mathbf{R}^\top\left({}^Ad - {}^G\mathbf{p}_L^\top\,{}_G^A\mathbf{R}^{\top A}\hat{\mathbf{n}} + {}^G\mathbf{p}_A^\top\,{}_G^A\mathbf{R}^{\top A}\hat{\mathbf{n}}\right)$$
$$+ {}_G^L\mathbf{R}\,{}_G^A\mathbf{R}^\top\left(-{}^A\hat{\mathbf{n}}\,{}^G\mathbf{p}_L^\top\,{}_G^A\mathbf{R}^\top + {}^A\hat{\mathbf{n}}\,{}^G\mathbf{p}_A^\top\,{}_G^A\mathbf{R}^\top\right) \quad (48)$$

$$\mathbf{H}_d = {}_G^L\mathbf{R}\,{}_G^A\mathbf{R}^{\top A}\mathbf{n} \quad (49)$$

$$\mathbf{H}_{\Pi1} = \frac{1}{{}^A\hat{d}}\left(\mathbf{I}_{3\times3} - {}^A\hat{\mathbf{n}}\,{}^A\hat{\mathbf{n}}^\top\right) \quad (50)$$

$$\mathbf{H}_{\Pi2} = {}^A\hat{\mathbf{n}}^\top \quad (51)$$

5 IMU to LiDAR State Transform

As of right now we have assumed that the IMU and LiDAR states have co-sided. This is never the case in practice due to having physical sensors. Our graph will stores IMU states since preintegration measurement require the state to be in the same "intertial" frame of reference as the acceleration and angular velocities, while all plane measurements are in the LiDAR frame of reference. We can transform from the IMU state to the LiDAR state as follows:

$${}_G^L\mathbf{R} = {}_I^L\mathbf{R}\,{}_G^I\mathbf{R} \quad (52)$$

$${}^G\mathbf{p}_L = {}^G\mathbf{p}_I - {}_G^I\mathbf{R}^\top\,{}_I^L\mathbf{R}^{\top L}\mathbf{p}_I \quad (53)$$

where ${}_I^L\mathbf{R}$ and ${}^L\mathbf{p}_I$ are the rotation from the IMU to LiDAR frame and the position of the IMU seen from the LiDAR frame, respectively. These could be calibrated online, but in this work we consider that theses transforms are known beforehand. We find the Jacobians of the above LiDAR states in respect the IMU state as:

$$\mathbf{H}_{IL} = \begin{bmatrix} \frac{\partial {}_G^L\tilde{\boldsymbol{\theta}}}{\partial {}_G^I\tilde{\boldsymbol{\theta}}} & \mathbf{0}_{3\times9} & \frac{\partial {}_G^L\tilde{\boldsymbol{\theta}}}{\partial {}^G\tilde{\mathbf{p}}_I} \\ \frac{\partial {}^G\tilde{\mathbf{p}}_L}{\partial {}_G^I\tilde{\boldsymbol{\theta}}} & \mathbf{0}_{3\times9} & \frac{\partial {}^G\tilde{\mathbf{p}}_L}{\partial {}^G\tilde{\mathbf{p}}_I} \end{bmatrix}_{6\times15} \quad (54)$$

$$= \begin{bmatrix} {}_I^L\mathbf{R} & \mathbf{0}_{3\times9} & \mathbf{0}_{3\times3} \\ {}_G^I\hat{\mathbf{R}}^\top\lfloor {}_I^L\mathbf{R}^{\top L}\mathbf{p}_I\times\rfloor & \mathbf{0}_{3\times9} & \mathbf{I}_{3\times3} \end{bmatrix}_{6\times15} \quad (55)$$

The derivations for the above equations can be found in Appendix E. This Jacobian should be chained with the $\mathbf{H}_L$ and $\mathbf{H}_A$ Jacobians from Section 4 to convert them to be functions of the estimated IMU states.

6 Initial Plane Graph Factor

We treat the first frame that a plane is measured from as its anchor. This first reading of the plane is a direct reading of the plane parameters we want to estimate, thus we can define the following factor:

$$^{A}\widetilde{\boldsymbol{\Pi}} = {}^{A}\widehat{\boldsymbol{\Pi}} - {}^{A}\boldsymbol{\Pi}_m \tag{56}$$

where $^{G}\boldsymbol{\Pi}_m$ is the measurement measured from the first frame and $^{A}\widehat{\boldsymbol{\Pi}}$ is the current estimated value of the plane. It is important to note that this measurement is only a function of the initial reading and define the Jacobians as the following:

$$\mathbf{H}_u = \mathbf{I}_{3\times3} \tag{57}$$

$$\mathbf{H}_x = \mathbf{0}_{3\times15} \tag{58}$$

7 Plane Correspondences

Outside of simulation, finding the correspondences between the planes presents a challenge. To overcome this, we calculate the Mahalanobis distance between a new local plane measurement and each of the currently estimated planes. If the minimal distance squared is smaller then a three degree-of-freedom chi-squared test then the two planes match.

1. First compute the joint covariance $\mathbf{P}_J$ of the anchor state, local state, and anchor plane.

$$\mathbf{P}_J = \text{gtsam.jointMarginalCovariance}(X(anchor), X(local), P(plane)) \tag{59}$$

2. Compute the residual covariance matrix of the current measurement given all previous measurements of this plane.

$$\mathbf{S} = \mathbf{H}\mathbf{P}_J\mathbf{H}^\top + \mathbf{R}_\Pi \tag{60}$$

3. Compute the error of the current measurement and the current estimate of that measurement transformed into the local frame:

$$^{L}\widetilde{\boldsymbol{\Pi}} = {}^{L}\boldsymbol{\Pi} - {}^{L}\widehat{\boldsymbol{\Pi}} \tag{61}$$

$$= \left({}^{L}_{A}\mathbf{R}^{A}\mathbf{n}\right)\left({}^{A}d - {}^{A}\mathbf{p}_L^\top{}^{A}\mathbf{n}\right) - {}^{L}\widehat{\boldsymbol{\Pi}} \tag{62}$$

4. Compute the Mahalanobis distance squared.

$$D = {}^{L}\widetilde{\boldsymbol{\Pi}}^\top \mathbf{S}^{-1}\ {}^{L}\widetilde{\boldsymbol{\Pi}} \tag{63}$$

5. After computing the above distance squared the minimum will be the best plane correspondence to the new measurement. To check if the plane is "close enough" to the currently estimation plane, threshold on a 95 percentile chi-squared distribution for a given three degrees of freedom.

$$\textbf{if}(D < \text{chi\_squared\_table}[3]) : \text{accept correspondence} \tag{64}$$

$$\textbf{else} : \text{measurement is of a new plane} \tag{65}$$

8 Continuous Preintegration

8.1 Inertial Measurement Model

We model the linear acceleration and angular velocity inertial measurements as:

$$\boldsymbol{\omega}_m = \boldsymbol{\omega} + \mathbf{b}_w + \mathbf{n}_w \tag{66}$$

$$\mathbf{a}_m = \mathbf{a} + \mathbf{b}_a + \mathbf{n}_a + {}_G^I\mathbf{R}^G\mathbf{g} \tag{67}$$

where ${}^G\mathbf{g}$ is the gravity in the global frame, $\boldsymbol{\omega}$ is the angular velocity, $\mathbf{a}$ is the linear acceleration, and $\mathbf{n}_w$, $\mathbf{n}_a$ are the continuous measurement noises. The underlying standard IMU dynamics are given by [3]:

$$ {}_G^I\dot{\bar{q}} = \frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega}_m - \mathbf{b}_w - \mathbf{n}_w){}_G^I\bar{q} \tag{68}$$

$$\dot{\mathbf{b}}_w = \mathbf{n}_{wb} \tag{69}$$

$$ {}^G\dot{\mathbf{v}}_k = {}_k^G\mathbf{R}\left(\mathbf{a}_m - \mathbf{b}_a - \mathbf{n}_a\right) - {}^G\mathbf{g} \tag{70}$$

$$\dot{\mathbf{b}}_a = \mathbf{n}_{ab} \tag{71}$$

$$ {}^G\dot{\mathbf{p}}_k = {}^G\mathbf{v}_k \tag{72}$$

where $\mathbf{n}_{wb}$, $\mathbf{n}_{ab}$ are the random walk noises and $\boldsymbol{\Omega}(\cdot)$ is:

$$\boldsymbol{\Omega}(\boldsymbol{\omega}) = \begin{bmatrix} -\lfloor\boldsymbol{\omega}\times\rfloor & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^\top & 0 \end{bmatrix} \tag{73}$$

8.2 Continuous Preintegrated Measurements

The key idea of continuous preintegration [4] is to factorize the resulting integration of equation (68)-(72) between two LiDAR timesteps:

$$ {}^G\mathbf{p}_{k+1} = {}^G\mathbf{p}_k + {}^G\mathbf{v}_k\Delta T - \frac{1}{2}{}^G\mathbf{g}\Delta T^2 + {}_k^G\mathbf{R}^k\boldsymbol{\alpha}_{k+1} \tag{74}$$

$$ {}^G\mathbf{v}_{k+1} = {}^G\mathbf{v}_k - {}^G\mathbf{g}\Delta T + {}_k^G\mathbf{R}^k\boldsymbol{\beta}_{k+1} \tag{75}$$

$$ {}_G^{k+1}\bar{q} = {}_k^{k+1}\bar{q} \otimes {}_G^k\bar{q} \tag{76}$$

where ΔT is the difference between the bounding LiDAR pose timestamps (t_k, t_{k+1}) and ${}^k\boldsymbol{\alpha}_{k+1}, {}^k\boldsymbol{\beta}_{k+1}$ are defined by the following integrations of the IMU measurements:

$$ {}^k\boldsymbol{\alpha}_{k+1} = \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} {}_u^k\mathbf{R}\left(\mathbf{a}_m - \mathbf{b}_a - \mathbf{n}_a\right) du\, ds \tag{77}$$

$$ {}^k\boldsymbol{\beta}_{k+1} = \int_{t_k}^{t_{k+1}} {}_u^k\mathbf{R}\left(\mathbf{a}_m - \mathbf{b}_a - \mathbf{n}_a\right) du \tag{78}$$

We note that the preintegrated measurements, ${}^k\boldsymbol{\alpha}_{k+1}, {}^k\boldsymbol{\beta}_{k+1}, {}_k^{k+1}\bar{q}$ are dependent on the *true* biases. This dependency is addressed through a first order Taylor series expansion about the current bias

estimates $\bar{\mathbf{b}}_w$ and $\bar{\mathbf{b}}_a$ at time t_k:

$$
{}^k\boldsymbol{\alpha}_{k+1} \simeq {}^k\breve{\boldsymbol{\alpha}}_{k+1} + \frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{b}_a}\bigg|_{\bar{\mathbf{b}}_a} \Delta \mathbf{b}_a + \frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{b}_w}\bigg|_{\bar{\mathbf{b}}_w} \Delta \mathbf{b}_w
\tag{79}
$$

$$
{}^k\boldsymbol{\beta}_{k+1} \simeq {}^k\breve{\boldsymbol{\beta}}_{k+1} + \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{b}_a}\bigg|_{\bar{\mathbf{b}}_a} \Delta \mathbf{b}_a + \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{b}_w}\bigg|_{\bar{\mathbf{b}}_w} \Delta \mathbf{b}_w
\tag{80}
$$

$$
{}^{k+1}_{k}\bar{q} \simeq \bar{q}(\Delta \mathbf{b}_w)^{-1} \otimes {}^{k+1}_{k}\breve{q}
\tag{81}
$$

where ${}^k\breve{\boldsymbol{\alpha}}_{k+1}$, ${}^k\breve{\boldsymbol{\beta}}_{k+1}$, ${}^{k+1}_{k}\breve{q}$ are the preintegrated measurements evaluated at the current bias estimates. In particular, ${}^{k+1}_{k}\breve{q}$ can be found using the zeroth order quaternion integrator [1]. We define the quaternion which models multiplicative orientation corrections due to changes in the linearized bias as:

$$
\bar{q}(\Delta \mathbf{b}_w) = \begin{bmatrix} \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|} \sin \frac{\|\boldsymbol{\theta}\|}{2} \\ \cos \frac{\|\boldsymbol{\theta}\|}{2} \end{bmatrix}
\tag{82}
$$

$$
\boldsymbol{\theta} = \frac{\partial \bar{q}}{\partial \mathbf{b}_w}\bigg|_{\bar{\mathbf{b}}_w} \left(\mathbf{b}_{w(k)} - \bar{\mathbf{b}}_w \right)
\tag{83}
$$

where $\Delta \mathbf{b}_w := \mathbf{b}_{w(k)} - \bar{\mathbf{b}}_w$ and $\Delta \mathbf{b}_a := \mathbf{b}_{a(k)} - \bar{\mathbf{b}}_a$ are the differences between the true biases and the current bias estimate used as the linearization point. The new preintegration measurements can now be computed *once* and changes in the bias estimates can be taken into account through the above Taylor series. The final measurement residual is as follows:

$$
r_I(\mathbf{x}) = \begin{bmatrix}
2\mathrm{vec}\left({}^{k+1}_{G}\bar{q} \otimes {}^k_G\bar{q}^{-1} \otimes {}^{k+1}_{k}\breve{q}^{-1} \otimes \bar{q}(\Delta \mathbf{b}_w) \right) \\[2mm]
\mathbf{b}_{w(k+1)} - \mathbf{b}_{w(k)} \\[2mm]
\left(\begin{array}{c} {}^k_G\mathbf{R}\left({}^G\mathbf{v}_{k+1} - {}^G\mathbf{v}_k + {}^G\mathbf{g}\Delta T \right) \\ -{}^k\breve{\boldsymbol{\beta}}_{k+1} - \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{b}_a}\big|_{\bar{\mathbf{b}}_a} \Delta \mathbf{b}_a - \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{b}_w}\big|_{\bar{\mathbf{b}}_w} \Delta \mathbf{b}_w \end{array} \right) \\[2mm]
\mathbf{b}_{a(k+1)} - \mathbf{b}_{a(k)} \\[2mm]
\left(\begin{array}{c} {}^k_G\mathbf{R}\left({}^G\mathbf{p}_{k+1} - {}^G\mathbf{p}_k - {}^G\mathbf{v}_k\Delta T + \frac{1}{2}{}^G\mathbf{g}\Delta T^2 \right) \\ -{}^k\breve{\boldsymbol{\alpha}}_{k+1} - \frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{b}_a}\big|_{\bar{\mathbf{b}}_a} \Delta \mathbf{b}_a - \frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{b}_w}\big|_{\bar{\mathbf{b}}_w} \Delta \mathbf{b}_w \end{array} \right)
\end{bmatrix}
$$

where $\mathrm{vec}(\cdot)$ returns the vector portion of the quaternion (i.e., the top three elements) and the bias errors are the difference between biases in the bounding states.

We use combined continuous preintegration factors that included both the inertial and bias errors together and relate to the full 15 degree-of-freedom state (see Equation 1). This combined continuous preintegration factor better models the measurement error state dynamics due to bias drift over the integration interval.

8.3 Continuous Preintegrated Jacobians

The analytical Jacobians needed for graph optimization, bias Jacobians, and *closed-form* preintegrated measurements are included in the preintegration technical report [5] where the above

Jacobians correspond to the following equations:

$$\left.\frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{b}_a}\right|_{\bar{\mathbf{b}}_a} = \text{Equation (49)} \tag{84}$$

$$\left.\frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{b}_w}\right|_{\bar{\mathbf{b}}_w} = \text{Equations (53)-(58),(84)} \tag{85}$$

$$\left.\frac{\partial \boldsymbol{\beta}}{\partial \mathbf{b}_a}\right|_{\bar{\mathbf{b}}_a} = \text{Equation (49)} \tag{86}$$

$$\left.\frac{\partial \boldsymbol{\beta}}{\partial \mathbf{b}_w}\right|_{\bar{\mathbf{b}}_w} = \text{Equations (59)-(61),(84)} \tag{87}$$

$$\left.\frac{\partial \bar{q}}{\partial \mathbf{b}_w}\right|_{\bar{\mathbf{b}}_w} = \text{Equation (81)} \tag{88}$$

8.4 Continuous Preintegrated Covariance

To find the covariance of the above residual, we can look at the continuous IMU error state dynamics. Consider the time $t_\tau \in [t_k, t_{k+1}]$. Defining $^k_\tau \breve{\mathbf{R}}$ as the rotation from the IMU frame at t_τ to the beginning IMU time t_k, $\hat{\mathbf{a}}$ as the corrected acceleration ($\hat{\mathbf{a}} = \mathbf{a}_m - \bar{\mathbf{b}}_a$), and $\hat{\boldsymbol{\omega}}$ as the corrected angular velocity ($\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}_m - \bar{\mathbf{b}}_\omega$), the linearized *measurement* error state system can be defined as the following:

$$\begin{bmatrix} ^\tau_k\dot{\delta\boldsymbol{\theta}} \\ \dot{\delta\mathbf{b}}_\omega \\ ^k\dot{\delta\boldsymbol{\beta}}_\tau \\ \dot{\delta\mathbf{b}}_a \\ ^k\dot{\delta\boldsymbol{\alpha}}_\tau \end{bmatrix} = \begin{bmatrix} -\lfloor\hat{\boldsymbol{\omega}}\times\rfloor & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -^k_\tau\breve{\mathbf{R}}\lfloor\hat{\mathbf{a}}\times\rfloor & \mathbf{0}_3 & \mathbf{0}_3 & -^k_\tau\breve{\mathbf{R}} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} \begin{bmatrix} ^\tau_k\delta\boldsymbol{\theta} \\ \delta\mathbf{b}_\omega \\ ^k\delta\boldsymbol{\beta}_\tau \\ \delta\mathbf{b}_a \\ ^k\delta\boldsymbol{\alpha}_\tau \end{bmatrix} + \begin{bmatrix} -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & -^k_\tau\breve{\mathbf{R}} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} \begin{bmatrix} \mathbf{n}_\omega \\ \mathbf{n}_{b\omega} \\ \mathbf{n}_a \\ \mathbf{n}_{a\omega} \end{bmatrix}$$

$$\Rightarrow \quad \dot{\mathbf{r}} = \mathbf{Fr} + \mathbf{Gn} \tag{89}$$

It can be noted that the above is equivalent to the standard VINS error state propagation equations in the local frame of reference [6]. Based on the above equations, we can define the state transition matrix $\boldsymbol{\Phi}(t_{\tau+1}, t_\tau)$ which describes how the error transitions across the measurement interval $[t_\tau, t_{\tau+1}] \subset [t_k, t_{k+1}]$. Starting with covariance $\mathbf{P}_k = \mathbf{0}_{15\times15}$ we perform the following propagation for all IMU measurements in the preintegration interval $[t_k, t_{k+1}]$.

$$\mathbf{P}_{\tau+1} = \boldsymbol{\Phi}(t_{\tau+1}, t_\tau)\mathbf{P}_\tau\boldsymbol{\Phi}(t_{\tau+1}, t_\tau)^\top + \mathbf{Q}_{p_\tau}$$

$$\mathbf{Q}_{p_\tau} = \int_{t_\tau}^{t_{\tau+1}} \boldsymbol{\Phi}(t_{\tau+1}, u)\mathbf{G}(u)\mathbf{Q}_c\mathbf{G}(u)^\top\boldsymbol{\Phi}(t_{\tau+1}, u)^\top du$$

where $\mathbf{Q}_c$ is the continuous time noise covariance matrix. The final covariance of the preintegrated measurement is the ending cumulative covariance $\mathbf{P}_{k+1}$.

Appendix A: Gauss-Newton Plane Weights

Our sensor provides noisy 3D points that lie on a known plane. We model the noise in these 3D points as the following:

$$^L\mathbf{p}_{mi} = {}^L\mathbf{p}_i + \mathbf{n}_p \tag{90}$$

$$\text{where } \mathbf{n}_p \sim \mathcal{N}(0, \mathbf{R}_d) \tag{91}$$

$$\mathbf{R}_d = \begin{bmatrix} \sigma_x^2 & 0 & \\ 0 & \sigma_y^2 & 0 \\ 0 & 0 & \sigma_z^2 \end{bmatrix}_{3 \times 3} \tag{92}$$

Using the above noisy measurement we can write the following measurement function:

$$\mathbf{z}_m = h({}^L\mathbf{\Pi}, \mathbf{n}_p) \tag{93}$$

Taking the Taylor series givens the following:

$$\mathbf{z}_m \simeq h({}^L\widehat{\mathbf{\Pi}}, \mathbf{n}_p) + \mathbf{J}_i \; {}^L\widetilde{\mathbf{\Pi}} + \mathbf{H}_n \; \mathbf{n}_p \tag{94}$$

Taking the expectation given the true state gives:

$$\mathbb{E}_{L_{\mathbf{\Pi}}}[\mathbf{z}_m] = \mathbb{E}_{L_{\mathbf{\Pi}}}\left[h({}^L\widehat{\mathbf{\Pi}}, \mathbf{n}_p) + \mathbf{J}_i \; {}^L\widetilde{\mathbf{\Pi}} + \mathbf{H}_n \; \mathbf{n}_p \right] \tag{95}$$

$$\hat{\mathbf{z}}_m = h({}^L\widehat{\mathbf{\Pi}}, \mathbf{n}_p) + \mathbf{J}_i \; {}^L\widetilde{\mathbf{\Pi}} \tag{96}$$

Finally we can calculate our covariance of the above measurement given the true state $^L\mathbf{\Pi}$:

$$\mathbb{E}_{L_{\mathbf{\Pi}}}\left[(\mathbf{z}_m - \hat{\mathbf{z}}_m)(\mathbf{z}_m - \hat{\mathbf{z}}_m)^\top \right] = \mathbb{E}_{L_{\mathbf{\Pi}}}\Big[\Big(h({}^L\widehat{\mathbf{\Pi}}, \mathbf{n}_p) + \mathbf{J}_i \; {}^L\widetilde{\mathbf{\Pi}} + \mathbf{H}_n \; \mathbf{n}_p$$
$$- h({}^L\widehat{\mathbf{\Pi}}, \mathbf{n}_p) - \mathbf{J}_i \; {}^L\widetilde{\mathbf{\Pi}}\Big)\Big(\cdots\Big)^\top\Big] \tag{97}$$

$$= \mathbb{E}_{L_{\mathbf{\Pi}}}\left[\Big(\mathbf{H}_n \; \mathbf{n}_p\Big)\Big(\mathbf{H}_n \; \mathbf{n}_p\Big)^\top\right] \tag{98}$$

$$= \mathbf{H}_n \; \mathbf{R}_d \; \mathbf{H}_n^\top \tag{99}$$

Giving us the following final equations:

$$W_i({}^L\widehat{\mathbf{\Pi}}) = \Big(\mathbf{H}_n \; \mathbf{R}_d\mathbf{H}_n^\top\Big)^{-1} \tag{100}$$

$$\mathbf{H}_n = -\frac{{}^L\widehat{\mathbf{\Pi}}^\top}{\left\| {}^L\widehat{\mathbf{\Pi}} \right\|} \tag{101}$$

Appendix B: Gradient of Gauss-Newton Cost Function

$$^{L}\widetilde{\mathbf{\Pi}}^{*} = \operatorname*{argmin}_{^{L}\widetilde{\mathbf{\Pi}}} \sum_{i=1}^{n} \left\| \, r_i(^{L}\widehat{\mathbf{\Pi}} \boxplus {}^{L}\widetilde{\mathbf{\Pi}}) \, \right\|_{W_i}^{2} \tag{102}$$

$$\simeq \operatorname*{argmin}_{^{L}\widetilde{\mathbf{\Pi}}} \sum_{i=1}^{n} \left\| \, r_i(^{L}\widehat{\mathbf{\Pi}}) + \mathbf{J}_i \, {}^{L}\widetilde{\mathbf{\Pi}} \, \right\|_{W_i}^{2} \tag{103}$$

$$\simeq \operatorname*{argmin}_{^{L}\widetilde{\mathbf{\Pi}}} \sum_{i=1}^{n} \left(r_i(^{L}\widehat{\mathbf{\Pi}}) + \mathbf{J}_i \, {}^{L}\widetilde{\mathbf{\Pi}} \right)^{\top} W_i \left(r_i(^{L}\widehat{\mathbf{\Pi}}) + \mathbf{J}_i \, {}^{L}\widetilde{\mathbf{\Pi}} \right) \tag{104}$$

$$\simeq \operatorname*{argmin}_{^{L}\widetilde{\mathbf{\Pi}}} \sum_{i=1}^{n} \left(r_i(^{L}\widehat{\mathbf{\Pi}})^{\top} W_i \, r_i(^{L}\widehat{\mathbf{\Pi}}) + r_i(^{L}\widehat{\mathbf{\Pi}})^{\top} W_i(\mathbf{J}_i \, {}^{L}\widetilde{\mathbf{\Pi}}) \right.$$
$$\left. + (\mathbf{J}_i \, {}^{L}\widetilde{\mathbf{\Pi}})^{\top} W_i \, r_i(^{L}\widehat{\mathbf{\Pi}}) + (\mathbf{J}_i \, {}^{L}\widetilde{\mathbf{\Pi}})^{\top} W_i(\mathbf{J}_i \, {}^{L}\widetilde{\mathbf{\Pi}}) \right) \tag{105}$$

$$\simeq \operatorname*{argmin}_{^{L}\widetilde{\mathbf{\Pi}}} \sum_{i=1}^{n} \left(r_i(^{L}\widehat{\mathbf{\Pi}})^{\top} W_i \, r_i(^{L}\widehat{\mathbf{\Pi}}) + 2 \, r_i(^{L}\widehat{\mathbf{\Pi}})^{\top} W_i(\mathbf{J}_i \, {}^{L}\widetilde{\mathbf{\Pi}}) \right.$$
$$\left. + (\mathbf{J}_i \, {}^{L}\widetilde{\mathbf{\Pi}})^{\top} W_i(\mathbf{J}_i \, {}^{L}\widetilde{\mathbf{\Pi}}) \right) \tag{106}$$

The above cost will be minimized when the gradient of the cost function is zero. Thus we take the gradient in respect to our optimization variable $\Delta^{L}\mathbf{\Pi}$:

$$\frac{\partial \, r_i(^{L}\widehat{\mathbf{\Pi}} \boxplus {}^{L}\widehat{\mathbf{\Pi}})}{\partial \, {}^{L}\widetilde{\mathbf{\Pi}}} = 0 + \sum_{i=1}^{n} \left(2 \, r_i(^{L}\widehat{\mathbf{\Pi}})^{\top} W_i \, \mathbf{J}_i \right)^{\top}$$
$$+ \sum_{i=1}^{n} (W_i \, \mathbf{J}_i)^{\top} \, \mathbf{J}_i \, {}^{L}\widetilde{\mathbf{\Pi}} + \sum_{i=1}^{n} \mathbf{J}_i^{\top}(W_i \, \mathbf{J}_i) \, {}^{L}\widetilde{\mathbf{\Pi}} \tag{107}$$

$$= \sum_{i=1}^{n} 2 \, (W_i \, \mathbf{J}_i)^{\top} r_i(^{L}\widehat{\mathbf{\Pi}})$$
$$+ \sum_{i=1}^{n} \left((W_i \, \mathbf{J}_i)^{\top} \, \mathbf{J}_i + \mathbf{J}_i^{\top}(W_i \, \mathbf{J}_i) \right) \, {}^{L}\widetilde{\mathbf{\Pi}} \tag{108}$$

$$= \sum_{i=1}^{n} 2 \, (W_i \, \mathbf{J}_i)^{\top} r_i(^{L}\widehat{\mathbf{\Pi}}) + \sum_{i=1}^{n} \left(2 \, (\mathbf{J}_i^{\top} \, W_i \, \mathbf{J}_i) \right) \, {}^{L}\widetilde{\mathbf{\Pi}} \tag{109}$$

$$-\sum_{i=1}^{n} \left(2 \, (\mathbf{J}_i^{\top} \, W_i \, \mathbf{J}_i) \right) \, {}^{L}\widetilde{\mathbf{\Pi}} = \sum_{i=1}^{n} 2 \, (W_i \, \mathbf{J}_i)^{\top} r_i(^{L}\widehat{\mathbf{\Pi}}) \tag{110}$$

$$^{L}\widetilde{\mathbf{\Pi}} = - \left(\sum_{i=1}^{n} 2 \, (\mathbf{J}_i^{\top} \, W_i \, \mathbf{J}_i) \right)^{-1} \left(\sum_{i=1}^{n} 2 \, (W_i \, \mathbf{J}_i)^{\top} r_i(^{L}\widehat{\mathbf{\Pi}}) \right) \tag{111}$$

$$^{L}\widetilde{\mathbf{\Pi}} = - \left(\sum_{i=1}^{n} \mathbf{J}_i^{\top} \, W_i \, \mathbf{J}_i \right)^{-1} \left(\sum_{i=1}^{n} \mathbf{J}_i^{\top} W_i \, r_i(^{L}\widehat{\mathbf{\Pi}}) \right) \tag{112}$$

Appendix C: Plane Measurement Compression Jacobian

Our sensor provides noisy 3D points that lie on a known plane. We model the noise in these 3D points as the following:

$$^{L}\mathbf{p}_{mi} = {}^{L}\mathbf{p}_i + \mathbf{n}_p \tag{113}$$

$$\text{where } \mathbf{n}_p \sim \mathcal{N}(0, \mathbf{R}_d) \tag{114}$$

$$\mathbf{R}_d = \begin{bmatrix} \sigma_x^2 & 0 & \\ 0 & \sigma_y^2 & 0 \\ 0 & 0 & \sigma_z^2 \end{bmatrix}_{3 \times 3} \tag{115}$$

Using the above noisy measurement we can write the following measurement function:

$$\mathbf{z}_m = h({}^{L}\mathbf{\Pi}, {}^{L}\mathbf{p}_i) \tag{116}$$

$$= \frac{{}^{L}\widehat{\mathbf{\Pi}}^{\top}}{\|{}^{L}\widehat{\mathbf{\Pi}}\|} {}^{L}\mathbf{p}_i - \|{}^{L}\widehat{\mathbf{\Pi}}\| \tag{117}$$

$$= \frac{{}^{L}\widehat{\mathbf{\Pi}}^{\top}}{\|{}^{L}\widehat{\mathbf{\Pi}}\|} \left({}^{L}\mathbf{p}_{mi} - \mathbf{n}_p\right) - \|{}^{L}\widehat{\mathbf{\Pi}}\| \tag{118}$$

$$= \frac{{}^{L}\widehat{\mathbf{\Pi}}^{\top}}{\|{}^{L}\widehat{\mathbf{\Pi}}\|} {}^{L}\mathbf{p}_{mi} - \|{}^{L}\widehat{\mathbf{\Pi}}\| - \frac{{}^{L}\widehat{\mathbf{\Pi}}^{\top}}{\|{}^{L}\widehat{\mathbf{\Pi}}\|} \mathbf{n}_p \tag{119}$$

We can take the the derivative of this function in respect to the error evaluating it with noises at zero and all values at the expected:

$$\mathbf{J}_i = \frac{\partial}{\partial^{L}\widetilde{\mathbf{\Pi}}} \left({}^{L}\mathbf{p}_{mi}^{\top} {}^{L}\widehat{\mathbf{\Pi}}\right) \left(\frac{1}{\|{}^{L}\widehat{\mathbf{\Pi}}\|}\right) + \left({}^{L}\mathbf{p}_{mi}^{\top} {}^{L}\widehat{\mathbf{\Pi}}\right) \frac{\partial}{\partial^{L}\widetilde{\mathbf{\Pi}}} \left(\frac{1}{\|{}^{L}\widehat{\mathbf{\Pi}}\|}\right) - \frac{\partial}{\partial^{L}\widetilde{\mathbf{\Pi}}} \left(\|{}^{L}\widehat{\mathbf{\Pi}}\|\right) \tag{120}$$

$$= \frac{{}^{L}\mathbf{p}_{mi}^{\top}}{\|{}^{L}\widehat{\mathbf{\Pi}}\|} - \left({}^{L}\mathbf{p}_{mi}^{\top} {}^{L}\mathbf{\Pi}\right) \frac{{}^{L}\widehat{\mathbf{\Pi}}^{\top}}{\|{}^{L}\widehat{\mathbf{\Pi}}\|^3} - \frac{{}^{L}\widehat{\mathbf{\Pi}}^{\top}}{\|{}^{L}\widehat{\mathbf{\Pi}}\|} \tag{121}$$

Appendix D: Anchor Plane Factor Jacobians

D.1: $\mathbf{H}_A$ Jacobian

$$^L\boldsymbol{\Pi} = {}^L_G\mathbf{R}\,{}^A_G\mathbf{R}^\top\left({}^A\mathbf{n}^A d - {}^A\mathbf{n}^G\mathbf{p}_L^\top\,{}^A_G\mathbf{R}^{\top A}\mathbf{n} + {}^A\mathbf{n}^G\mathbf{p}_A^\top\,{}^A_G\mathbf{R}^{\top A}\mathbf{n}\right) \tag{122}$$

We perturb the rotation as follows:

$$^L\hat{\boldsymbol{\Pi}} + {}^L\tilde{\boldsymbol{\Pi}} = {}^L_G\mathbf{R}\left(\left(\mathbf{I} - \lfloor\delta\boldsymbol{\theta}\times\rfloor\right){}^A_G\hat{\mathbf{R}}\right)^\top$$

$$\left({}^A\mathbf{n}^A d - {}^A\mathbf{n}^G\mathbf{p}_L^\top\left(\left(\mathbf{I} - \lfloor\delta\boldsymbol{\theta}\times\rfloor\right){}^A_G\hat{\mathbf{R}}\right)^{\top A}\mathbf{n} + {}^A\mathbf{n}^G\mathbf{p}_A^\top\left(\left(\mathbf{I} - \lfloor\delta\boldsymbol{\theta}\times\rfloor\right){}^A_G\hat{\mathbf{R}}\right)^{\top A}\mathbf{n}\right) \tag{123}$$

$$= {}^L_G\mathbf{R}\,{}^A_G\hat{\mathbf{R}}^\top\left(\mathbf{I} + \lfloor\delta\boldsymbol{\theta}\times\rfloor\right)$$

$$\left({}^A\mathbf{n}^A d - {}^A\mathbf{n}^G\mathbf{p}_L^\top\,{}^A_G\hat{\mathbf{R}}^\top\left(\mathbf{I} + \lfloor\delta\boldsymbol{\theta}\times\rfloor\right){}^A\mathbf{n} + {}^A\mathbf{n}^G\mathbf{p}_A^\top\,{}^A_G\hat{\mathbf{R}}^\top\left(\mathbf{I} + \lfloor\delta\boldsymbol{\theta}\times\rfloor\right){}^A\mathbf{n}\right) \tag{124}$$

$$= {}^L_G\mathbf{R}\,{}^A_G\hat{\mathbf{R}}^\top\left(\mathbf{I} + \lfloor\delta\boldsymbol{\theta}\times\rfloor\right)$$

$$\left({}^A\mathbf{n}^A d - {}^A\mathbf{n}^G\mathbf{p}_L^\top\,{}^A_G\hat{\mathbf{R}}^{\top A}\mathbf{n} - {}^A\mathbf{n}^G\mathbf{p}_L^\top\,{}^A_G\hat{\mathbf{R}}^\top\lfloor\delta\boldsymbol{\theta}\times\rfloor{}^A\mathbf{n}\right.$$

$$\left. + {}^A\mathbf{n}^G\mathbf{p}_A^\top\,{}^A_G\hat{\mathbf{R}}^{\top A}\mathbf{n} + {}^A\mathbf{n}^G\mathbf{p}_A^\top\,{}^A_G\hat{\mathbf{R}}^\top\lfloor\delta\boldsymbol{\theta}\times\rfloor{}^A\mathbf{n}\right) \tag{125}$$

$$= {}^L_G\mathbf{R}\,{}^A_G\hat{\mathbf{R}}^\top$$

$$\left({}^A\mathbf{n}^A d - {}^A\mathbf{n}^G\mathbf{p}_L^\top\,{}^A_G\hat{\mathbf{R}}^{\top A}\mathbf{n} - {}^A\mathbf{n}^G\mathbf{p}_L^\top\,{}^A_G\hat{\mathbf{R}}^\top\lfloor\delta\boldsymbol{\theta}\times\rfloor{}^A\mathbf{n}\right.$$

$$+ {}^A\mathbf{n}^G\mathbf{p}_A^\top\,{}^A_G\hat{\mathbf{R}}^{\top A}\mathbf{n} + {}^A\mathbf{n}^G\mathbf{p}_A^\top\,{}^A_G\hat{\mathbf{R}}^\top\lfloor\delta\boldsymbol{\theta}\times\rfloor{}^A\mathbf{n}$$

$$\left. + \lfloor\delta\boldsymbol{\theta}\times\rfloor{}^A\mathbf{n}^A d - \lfloor\delta\boldsymbol{\theta}\times\rfloor{}^A\mathbf{n}^G\mathbf{p}_L^\top\,{}^A_G\hat{\mathbf{R}}^{\top A}\mathbf{n} + \lfloor\delta\boldsymbol{\theta}\times\rfloor{}^A\mathbf{n}^G\mathbf{p}_A^\top\,{}^A_G\hat{\mathbf{R}}^{\top A}\mathbf{n}\right) \tag{126}$$

$$= {}^L\hat{\boldsymbol{\Pi}} + {}^L_G\mathbf{R}\,{}^A_G\hat{\mathbf{R}}^\top\left(-{}^A\mathbf{n}^G\mathbf{p}_L^\top\,{}^A_G\hat{\mathbf{R}}^\top\lfloor\delta\boldsymbol{\theta}\times\rfloor{}^A\mathbf{n} + {}^A\mathbf{n}^G\mathbf{p}_A^\top\,{}^A_G\hat{\mathbf{R}}^\top\lfloor\delta\boldsymbol{\theta}\times\rfloor{}^A\mathbf{n}\right.$$

$$\left. + \lfloor\delta\boldsymbol{\theta}\times\rfloor{}^A\mathbf{n}^A d - \lfloor\delta\boldsymbol{\theta}\times\rfloor{}^A\mathbf{n}^G\mathbf{p}_L^\top\,{}^A_G\hat{\mathbf{R}}^{\top A}\mathbf{n} + \lfloor\delta\boldsymbol{\theta}\times\rfloor{}^A\mathbf{n}^G\mathbf{p}_A^\top\,{}^A_G\hat{\mathbf{R}}^{\top A}\mathbf{n}\right) \tag{127}$$

$$= {}^L\hat{\boldsymbol{\Pi}} + {}^L_G\mathbf{R}\,{}^A_G\hat{\mathbf{R}}^\top\left({}^A\mathbf{n}^G\mathbf{p}_L^\top\,{}^A_G\hat{\mathbf{R}}^\top\lfloor{}^A\mathbf{n}\times\rfloor - {}^A\mathbf{n}^G\mathbf{p}_A^\top\,{}^A_G\hat{\mathbf{R}}^\top\lfloor{}^A\mathbf{n}\times\rfloor\right.$$

$$\left. - \lfloor{}^A\mathbf{n}^A d\times\rfloor + \lfloor{}^A\mathbf{n}^G\mathbf{p}_L^\top\,{}^A_G\hat{\mathbf{R}}^{\top A}\mathbf{n}\times\rfloor - \lfloor{}^A\mathbf{n}^G\mathbf{p}_A^\top\,{}^A_G\hat{\mathbf{R}}^{\top A}\mathbf{n}\times\rfloor\right)\delta\boldsymbol{\theta} \tag{128}$$

Thus, we have the following:

$$\boxed{\frac{\partial\, h}{\partial\, {}_{G}^{A}\boldsymbol{\delta\theta}} = {}_{G}^{L}\mathbf{R}\, {}_{G}^{A}\hat{\mathbf{R}}^{\top}\left({}^{A}\mathbf{n}\, {}^{G}\mathbf{p}_{L}^{\top}\, {}_{G}^{A}\hat{\mathbf{R}}^{\top}\lfloor {}^{A}\mathbf{n}\,\times\rfloor - {}^{A}\mathbf{n}\, {}^{G}\mathbf{p}_{A}^{\top}\, {}_{G}^{A}\hat{\mathbf{R}}^{\top}\lfloor {}^{A}\mathbf{n}\,\times\rfloor\right.}$$

$$\boxed{\left.- \lfloor {}^{A}\mathbf{n}\, {}^{A}d\,\times\rfloor + \left\lfloor {}^{A}\mathbf{n}\, {}^{G}\mathbf{p}_{L}^{\top}\, {}_{G}^{A}\hat{\mathbf{R}}^{\top\, A}\mathbf{n}\,\times\right\rfloor - \left\lfloor {}^{A}\mathbf{n}\, {}^{G}\mathbf{p}_{A}^{\top}\, {}_{G}^{A}\hat{\mathbf{R}}^{\top\, A}\mathbf{n}\,\times\right\rfloor\right)} \tag{129}$$

We perturb the position as follows:

$${}^{L}\hat{\boldsymbol{\Pi}} + {}^{L}\tilde{\boldsymbol{\Pi}} = {}_{G}^{L}\mathbf{R}\, {}_{G}^{A}\mathbf{R}^{\top}\left({}^{A}\mathbf{n}\, {}^{A}d - {}^{A}\mathbf{n}\, {}^{G}\mathbf{p}_{L\,G}^{\top\,A}\mathbf{R}^{\top\,A}\mathbf{n} + {}^{A}\mathbf{n}({}^{G}\hat{\mathbf{p}}_{A} + {}^{G}\tilde{\mathbf{p}}_{A})^{\top}\, {}_{G}^{A}\mathbf{R}^{\top\,A}\mathbf{n}\right) \tag{130}$$

$$= {}_{G}^{L}\mathbf{R}\, {}_{G}^{A}\mathbf{R}^{\top}\left({}^{A}\mathbf{n}\, {}^{A}d - {}^{A}\mathbf{n}\, {}^{G}\mathbf{p}_{L}^{\top}\, {}_{G}^{A}\mathbf{R}^{\top\,A}\mathbf{n} + {}^{A}\mathbf{n}\, {}^{G}\hat{\mathbf{p}}_{A}^{\top}\, {}_{G}^{A}\mathbf{R}^{\top\,A}\mathbf{n} + {}^{A}\mathbf{n}\, {}^{G}\tilde{\mathbf{p}}_{A}^{\top}\, {}_{G}^{A}\mathbf{R}^{\top\,A}\mathbf{n}\right) \tag{131}$$

$$= {}^{L}\hat{\boldsymbol{\Pi}} + {}_{G}^{L}\mathbf{R}\, {}_{G}^{A}\mathbf{R}^{\top}\left({}^{A}\mathbf{n}\, {}^{G}\tilde{\mathbf{p}}_{A}^{\top}\, {}_{G}^{A}\mathbf{R}^{\top\,A}\mathbf{n}\right) \tag{132}$$

$$= {}^{L}\hat{\boldsymbol{\Pi}} + {}_{G}^{L}\mathbf{R}\, {}_{G}^{A}\mathbf{R}^{\top}\left({}^{A}\mathbf{n}\,({}_{G}^{A}\mathbf{R}^{\top\,A}\mathbf{n})^{\top}\right){}^{G}\tilde{\mathbf{p}}_{A} \tag{133}$$

$$= {}^{L}\hat{\boldsymbol{\Pi}} + \left({}_{G}^{L}\mathbf{R}\, {}_{G}^{A}\mathbf{R}^{\top\,A}\mathbf{n}\, {}^{A}\mathbf{n}^{\top}\, {}_{G}^{A}\mathbf{R}\right){}^{G}\tilde{\mathbf{p}}_{A} \tag{134}$$

Thus, we have the following:

$$\boxed{\frac{\partial\, h}{\partial\, {}^{G}\tilde{\mathbf{p}}_{A}} = {}_{G}^{L}\mathbf{R}\, {}_{G}^{A}\mathbf{R}^{\top\,A}\mathbf{n}\, {}^{A}\mathbf{n}^{\top}\, {}_{G}^{A}\mathbf{R}} \tag{135}$$

D.2: $\mathbf{H}_{L}$ Jacobian

$${}^{L}\boldsymbol{\Pi} = {}_{G}^{L}\mathbf{R}\, {}_{G}^{A}\mathbf{R}^{\top}\left({}^{A}\mathbf{n}\, {}^{A}d - {}^{A}\mathbf{n}\, {}^{G}\mathbf{p}_{L}^{\top}\, {}_{G}^{A}\mathbf{R}^{\top\,A}\mathbf{n} + {}^{A}\mathbf{n}\, {}^{G}\mathbf{p}_{A}^{\top}\, {}_{G}^{A}\mathbf{R}^{\top\,A}\mathbf{n}\right) \tag{136}$$

We perturb the rotation as follows:

$${}^{L}\hat{\boldsymbol{\Pi}} + {}^{L}\tilde{\boldsymbol{\Pi}} = \left(\mathbf{I} - \lfloor\boldsymbol{\delta\theta}\times\rfloor\right){}_{G}^{L}\hat{\mathbf{R}}\, {}_{G}^{A}\mathbf{R}^{\top}\left(\cdots\right) \tag{137}$$

$$= {}^{L}\hat{\boldsymbol{\Pi}} - \lfloor\boldsymbol{\delta\theta}\times\rfloor\, {}_{G}^{L}\hat{\mathbf{R}}\, {}_{G}^{A}\mathbf{R}^{\top}\left(\cdots\right) \tag{138}$$

$$= {}^{L}\hat{\boldsymbol{\Pi}} + \left\lfloor {}_{G}^{L}\hat{\mathbf{R}}\, {}_{G}^{A}\mathbf{R}^{\top}\left(\cdots\right)\times\right\rfloor\boldsymbol{\delta\theta} \tag{139}$$

Thus, we have the following:

$$\boxed{\frac{\partial\, h}{\partial\, {}_{G}^{L}\boldsymbol{\delta\theta}} = \left\lfloor {}_{G}^{L}\hat{\mathbf{R}}\, {}_{G}^{A}\mathbf{R}^{\top}\left({}^{A}\mathbf{n}\, {}^{A}d - {}^{A}\mathbf{n}\, {}^{G}\mathbf{p}_{L}^{\top}\, {}_{G}^{A}\mathbf{R}^{\top\,A}\mathbf{n} + {}^{A}\mathbf{n}\, {}^{G}\mathbf{p}_{A}^{\top}\, {}_{G}^{A}\mathbf{R}^{\top\,A}\mathbf{n}\right)\times\right\rfloor} \tag{140}$$

We perturb the position as follows:

$$
{}^L\hat{\boldsymbol{\Pi}} + {}^L\tilde{\boldsymbol{\Pi}} = {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^\top \left({}^A\mathbf{n}^A d - {}^A\mathbf{n}({}^G\hat{\mathbf{p}}_L + {}^G\tilde{\mathbf{p}}_L)^\top\, {}^A_G\mathbf{R}^{\top A}\mathbf{n} + {}^A\mathbf{n}^G\mathbf{p}_A^\top\, {}^A_G\mathbf{R}^{\top A}\mathbf{n} \right) \tag{141}
$$

$$
= {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^\top \left({}^A\mathbf{n}^A d - {}^A\mathbf{n}^G\hat{\mathbf{p}}_L^\top\, {}^A_G\mathbf{R}^{\top A}\mathbf{n} - {}^A\mathbf{n}^G\tilde{\mathbf{p}}_L^\top\, {}^A_G\mathbf{R}^{\top A}\mathbf{n} + {}^A\mathbf{n}^G\mathbf{p}_A^\top\, {}^A_G\mathbf{R}^{\top A}\mathbf{n} \right) \tag{142}
$$

$$
= {}^L\hat{\boldsymbol{\Pi}} + {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^\top \left(- {}^A\mathbf{n}^G\tilde{\mathbf{p}}_L^\top\, {}^A_G\mathbf{R}^{\top A}\mathbf{n} \right) \tag{143}
$$

$$
= {}^L\hat{\boldsymbol{\Pi}} + {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^\top \left(- {}^A\mathbf{n}\, ({}^A_G\mathbf{R}^{\top A}\mathbf{n})^\top \right)^G\tilde{\mathbf{p}}_L \tag{144}
$$

$$
= {}^L\hat{\boldsymbol{\Pi}} + \left(- {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^{\top A}\mathbf{n}\, {}^A\mathbf{n}^{\top A}_G\mathbf{R} \right)^G\tilde{\mathbf{p}}_L \tag{145}
$$

Thus, we have the following:

$$
\boxed{\frac{\partial\, h}{\partial\, {}^G\tilde{\mathbf{p}}_L} = - {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^{\top A}\mathbf{n}\, {}^A\mathbf{n}^{\top A}_G\mathbf{R}} \tag{146}
$$

D.3: $\mathbf{H}_{nd}$ Jacobian

$$
{}^L\boldsymbol{\Pi} = {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^\top \left({}^A\mathbf{n}^A d - {}^A\mathbf{n}^G\mathbf{p}_L^\top\, {}^A_G\mathbf{R}^{\top A}\mathbf{n} + {}^A\mathbf{n}^G\mathbf{p}_A^\top\, {}^A_G\mathbf{R}^{\top A}\mathbf{n} \right) \tag{147}
$$

We perturb the normal vector ${}^A\mathbf{n}$ as follows:

$$
\begin{aligned}
{}^L\hat{\boldsymbol{\Pi}} + {}^L\tilde{\boldsymbol{\Pi}} = {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^\top \Big(& ({}^A\hat{\mathbf{n}} + {}^A\tilde{\mathbf{n}})^A d - ({}^A\hat{\mathbf{n}} + {}^A\tilde{\mathbf{n}})^G\mathbf{p}_L^\top\, {}^A_G\mathbf{R}^\top({}^A\hat{\mathbf{n}} + {}^A\tilde{\mathbf{n}}) \\
& + ({}^A\hat{\mathbf{n}} + {}^A\tilde{\mathbf{n}})^G\mathbf{p}_A^\top\, {}^A_G\mathbf{R}^\top({}^A\hat{\mathbf{n}} + {}^A\tilde{\mathbf{n}}) \Big)
\end{aligned} \tag{148}
$$

$$
\begin{aligned}
= {}^L\hat{\boldsymbol{\Pi}} + {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^\top \Big(& {}^A\tilde{\mathbf{n}}^A d - {}^A\hat{\mathbf{n}}^G\mathbf{p}_L^\top\, {}^A_G\mathbf{R}^{\top A}\tilde{\mathbf{n}} - {}^A\tilde{\mathbf{n}}^G\mathbf{p}_L^\top\, {}^A_G\mathbf{R}^{\top A}\hat{\mathbf{n}} \\
& + {}^A\hat{\mathbf{n}}^G\mathbf{p}_A^\top\, {}^A_G\mathbf{R}^{\top A}\tilde{\mathbf{n}} + {}^A\tilde{\mathbf{n}}^G\mathbf{p}_A^\top\, {}^A_G\mathbf{R}^{\top A}\hat{\mathbf{n}} \Big)
\end{aligned} \tag{149}
$$

$$
\begin{aligned}
= {}^L\hat{\boldsymbol{\Pi}} + {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^\top \Big(& {}^A d - {}^A\hat{\mathbf{n}}^G\mathbf{p}_L^\top\, {}^A_G\mathbf{R}^\top - {}^G\mathbf{p}_L^\top\, {}^A_G\mathbf{R}^{\top A}\hat{\mathbf{n}} \\
& + {}^A\hat{\mathbf{n}}^G\mathbf{p}_A^\top\, {}^A_G\mathbf{R}^\top + {}^G\mathbf{p}_A^\top\, {}^A_G\mathbf{R}^{\top A}\hat{\mathbf{n}} \Big)^A\tilde{\mathbf{n}}
\end{aligned} \tag{150}
$$

Thus, we have the following:

$$
\boxed{\begin{aligned}
\frac{\partial\, h}{\partial\, {}^A\tilde{\mathbf{n}}} & = {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^\top \left({}^A d - {}^A\hat{\mathbf{n}}^G\mathbf{p}_L^\top\, {}^A_G\mathbf{R}^\top - {}^G\mathbf{p}_L^\top\, {}^A_G\mathbf{R}^{\top A}\hat{\mathbf{n}} + {}^A\hat{\mathbf{n}}^G\mathbf{p}_A^\top\, {}^A_G\mathbf{R}^\top + {}^G\mathbf{p}_A^\top\, {}^A_G\mathbf{R}^{\top A}\hat{\mathbf{n}} \right) \\
& = {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^\top \left({}^A d - {}^G\mathbf{p}_L^\top\, {}^A_G\mathbf{R}^{\top A}\hat{\mathbf{n}} + {}^G\mathbf{p}_A^\top\, {}^A_G\mathbf{R}^{\top A}\hat{\mathbf{n}} \right) \\
& \qquad\qquad + {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^\top \left(- {}^A\hat{\mathbf{n}}^G\mathbf{p}_L^\top\, {}^A_G\mathbf{R}^\top + {}^A\hat{\mathbf{n}}^G\mathbf{p}_A^\top\, {}^A_G\mathbf{R}^\top \right)
\end{aligned}} \tag{151}
$$

$$
\tag{152}
$$

We perturb the distance ${}^A d$ as follows:

$$
{}^L\hat{\boldsymbol{\Pi}} + {}^L\tilde{\boldsymbol{\Pi}} = {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^\top \left({}^A\mathbf{n}({}^A\hat{d} + {}^A\tilde{d}) - {}^A\mathbf{n}^G\mathbf{p}_L^\top\, {}^A_G\mathbf{R}^{\top A}\mathbf{n} + {}^A\mathbf{n}^G\mathbf{p}_A^\top\, {}^A_G\mathbf{R}^{\top A}\mathbf{n} \right) \tag{153}
$$

$$
= {}^L\hat{\boldsymbol{\Pi}} + \left({}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^{\top A}\mathbf{n} \right)^A\tilde{d} \tag{154}
$$

$$
\tag{155}
$$

Thus, we have the following:

$$\frac{\partial\, h}{\partial\, ^A\tilde{d}} = {}^L_G\mathbf{R}\, {}^A_G\mathbf{R}^\top {}^A\mathbf{n} \tag{156}$$

D.4: $\mathbf{H}_\Pi$ Jacobian

$$\begin{bmatrix} ^A\mathbf{n} \\ ^A d \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{^A\Pi(x)^2 + ^A\Pi(y)^2 + ^A\Pi(z)^2}}\, {}^A\mathbf{\Pi} \\ \sqrt{^A\Pi(x)^2 + ^A\Pi(y)^2 + ^A\Pi(z)^2} \end{bmatrix} \tag{157}$$

We can take the element wise derivative to get the following:

$$\frac{\partial\, ^A\mathbf{n}}{\partial\, ^A\widetilde{\mathbf{\Pi}}} = \begin{bmatrix} \dfrac{^A\Pi(y)^2 + ^A\Pi(z)^2}{(^A\Pi(x)^2 + ^A\Pi(y)^2 + ^A\Pi(z)^2)^{\frac{3}{2}}} & -\dfrac{^A\Pi(x)^A\Pi(y)}{(^A\Pi(x)^2 + ^A\Pi(y)^2 + ^A\Pi(z)^2)^{\frac{3}{2}}} & -\dfrac{^A\Pi(x)^A\Pi(z)}{(^A\Pi(x)^2 + ^A\Pi(y)^2 + ^A\Pi(z)^2)^{\frac{3}{2}}} \\[4mm] -\dfrac{^A\Pi(y)^A\Pi(x)}{(^A\Pi(x)^2 + ^A\Pi(y)^2 + ^A\Pi(z)^2)^{\frac{3}{2}}} & \dfrac{^A\Pi(x)^2 + ^A\Pi(z)^2}{(^A\Pi(x)^2 + ^A\Pi(y)^2 + ^A\Pi(z)^2)^{\frac{3}{2}}} & -\dfrac{^A\Pi(y)^A\Pi(z)}{(^A\Pi(x)^2 + ^A\Pi(y)^2 + ^A\Pi(z)^2)^{\frac{3}{2}}} \\[4mm] -\dfrac{^A\Pi(z)^A\Pi(x)}{(^A\Pi(x)^2 + ^A\Pi(y)^2 + ^A\Pi(z)^2)^{\frac{3}{2}}} & -\dfrac{^A\Pi(z)^A\Pi(y)}{(^A\Pi(x)^2 + ^A\Pi(y)^2 + ^A\Pi(z)^2)^{\frac{3}{2}}} & \dfrac{^A\Pi(x)^2 + ^A\Pi(y)^2}{(^A\Pi(x)^2 + ^A\Pi(y)^2 + ^A\Pi(z)^2)^{\frac{3}{2}}} \end{bmatrix}$$

$$\frac{\partial\, ^A\mathbf{n}}{\partial\, ^A\widetilde{\mathbf{\Pi}}} = \frac{1}{\left(^A\Pi(x)^2 + ^A\Pi(y)^2 + ^A\Pi(z)^2\right)^{\frac{3}{2}}} \begin{bmatrix} ^A\Pi(y)^2 + ^A\Pi(z)^2 & -^A\Pi(x)^A\Pi(y) & -^A\Pi(x)^A\Pi(z) \\[3mm] -^A\Pi(y)^A\Pi(x) & ^A\Pi(x)^2 + ^A\Pi(z)^2 & -^A\Pi(y)^A\Pi(z) \\[3mm] -^A\Pi(z)^A\Pi(x) & -^A\Pi(z)^A\Pi(y) & ^A\Pi(x)^2 + ^A\Pi(y)^2 \end{bmatrix}$$

$$= \frac{1}{^A\hat{d}^3} \left(\begin{bmatrix} ^A\hat{d}^2 & 0 & 0 \\[2mm] 0 & ^A\hat{d}^2 & 0 \\[2mm] 0 & 0 & ^A\hat{d}^2 \end{bmatrix} - \begin{bmatrix} ^A\Pi(x)^2 & ^A\Pi(x)^A\Pi(y) & ^A\Pi(x)^A\Pi(z) \\[2mm] ^A\Pi(y)^A\Pi(x) & ^A\Pi(y)^2 & ^A\Pi(y)^A\Pi(z) \\[2mm] ^A\Pi(z)^A\Pi(x) & ^A\Pi(z)^A\Pi(y) & ^A\Pi(z)^2 \end{bmatrix} \right)$$

$$= \frac{1}{^A\hat{d}} \left(\mathbf{I}_{3\times3} - {}^A\hat{\mathbf{n}}\, {}^A\hat{\mathbf{n}}^\top \right)$$

$$\frac{\partial\, ^A\mathbf{n}}{\partial\, ^A\widetilde{\mathbf{\Pi}}} = \frac{1}{^A\hat{d}} \left(\mathbf{I}_{3\times3} - {}^A\hat{\mathbf{n}}\, {}^A\hat{\mathbf{n}}^\top \right) \tag{158}$$

We can then take the element wise derivatives in respect to the distance scalar.

$$\frac{\partial\, ^A d}{\partial\, ^A\widetilde{\mathbf{\Pi}}} = \frac{1}{\sqrt{^A\Pi(x)^2 + ^A\Pi(y)^2 + ^A\Pi(z)^2}} \begin{bmatrix} ^A\Pi(x) & ^A\Pi(y) & ^A\Pi(z) \end{bmatrix} \tag{159}$$

$$= {}^A\hat{\mathbf{n}}^\top \tag{160}$$

$$\frac{\partial\, ^A d}{\partial\, ^A\widetilde{\mathbf{\Pi}}} = {}^A\hat{\mathbf{n}}^\top \tag{161}$$

Appendix E: IMU to LIDAR Transform Jacobian

E.1: $\partial_G^L \tilde{\boldsymbol{\theta}}$ Jacobian

$$_G^L\mathbf{R} = {}_I^L\mathbf{R} \, {}_G^I\mathbf{R} \tag{162}$$

We perturb the rotation as follows:

$$\left(\mathbf{I} - \lfloor \delta\boldsymbol{\theta}\times \rfloor\right) {}_G^L\hat{\mathbf{R}} = {}_I^L\mathbf{R} \left(\mathbf{I} - \lfloor \delta\boldsymbol{\theta}\times \rfloor\right) {}_G^I\hat{\mathbf{R}} \tag{163}$$

$$= {}_I^L\mathbf{R} \, {}_G^I\hat{\mathbf{R}} - {}_I^L\mathbf{R} \, \lfloor \delta\boldsymbol{\theta}\times \rfloor {}_G^I\hat{\mathbf{R}} \tag{164}$$

$$= {}_I^L\mathbf{R} \, {}_G^I\hat{\mathbf{R}} - {}_I^L\mathbf{R} \, \lfloor \delta\boldsymbol{\theta}\times \rfloor {}_I^L\mathbf{R}^\top \, {}_I^L\mathbf{R} \, {}_G^I\hat{\mathbf{R}} \tag{165}$$

$$= \left(\mathbf{I} - \lfloor {}_I^L\mathbf{R} \, \delta\boldsymbol{\theta}\times \rfloor\right) {}_G^L\hat{\mathbf{R}} \tag{166}$$

Thus, we have the following:

$$\boxed{\frac{\partial_G^L \tilde{\boldsymbol{\theta}}}{\partial_G^I \tilde{\boldsymbol{\theta}}} = {}_I^L\mathbf{R}} \tag{167}$$

$$\boxed{\frac{\partial_G^L \tilde{\boldsymbol{\theta}}}{\partial^G \tilde{\mathbf{p}}_I} = \mathbf{0}_{3\times3}} \tag{168}$$

E.2: $\partial^G \tilde{\mathbf{p}}_L$ Jacobian

$$^G\mathbf{p}_L = {}^G\mathbf{p}_I - {}_G^I\mathbf{R}^\top \, {}_I^L\mathbf{R}^\top \, {}^L\mathbf{p}_I \tag{169}$$

We perturb the rotation as follows:

$$^G\hat{\mathbf{p}}_L + {}^G\tilde{\mathbf{p}}_L = {}^G\mathbf{p}_I - \left(\left(\mathbf{I} - \lfloor \delta\boldsymbol{\theta}\times \rfloor\right) {}_G^I\hat{\mathbf{R}}\right)^\top {}_I^L\mathbf{R}^\top \, {}^L\mathbf{p}_I \tag{170}$$

$$= {}^G\mathbf{p}_I - {}_G^I\hat{\mathbf{R}}^\top \left(\mathbf{I} + \lfloor \delta\boldsymbol{\theta}\times \rfloor\right) {}_I^L\mathbf{R}^\top \, {}^L\mathbf{p}_I \tag{171}$$

$$= {}^G\hat{\mathbf{p}}_L - {}_G^I\hat{\mathbf{R}}^\top \lfloor \delta\boldsymbol{\theta}\times \rfloor \, {}_I^L\mathbf{R}^\top \, {}^L\mathbf{p}_I \tag{172}$$

$$= {}^G\hat{\mathbf{p}}_L + {}_G^I\hat{\mathbf{R}}^\top \left\lfloor {}_I^L\mathbf{R}^\top \, {}^L\mathbf{p}_I \times \right\rfloor \delta\boldsymbol{\theta} \tag{173}$$

$$\boxed{\frac{\partial^G \tilde{\mathbf{p}}_L}{\partial_G^I \tilde{\boldsymbol{\theta}}} = {}_G^I\hat{\mathbf{R}}^\top \left\lfloor {}_I^L\mathbf{R}^\top \, {}^L\mathbf{p}_I \times \right\rfloor} \tag{174}$$

We perturb the position as follows:

$$^G\hat{\mathbf{p}}_L + {}^G\tilde{\mathbf{p}}_L = {}^G\hat{\mathbf{p}}_I + {}^G\tilde{\mathbf{p}}_I - {}_G^I\mathbf{R}^\top \, {}_I^L\mathbf{R}^\top \, {}^L\mathbf{p}_I \tag{175}$$

$$= {}^G\hat{\mathbf{p}}_L + {}^G\tilde{\mathbf{p}}_I \tag{176}$$

$$\boxed{\frac{\partial^G \tilde{\mathbf{p}}_L}{\partial^G \tilde{\mathbf{p}}_I} = \mathbf{I}_{3\times3}} \tag{177}$$

Appendix F: Analytic Jacobians for Quaternion Representation

F.1: Quaternion Representation

The plane can also be represented by the a quaternion $\bar{q}_\pi$, and the relation with the plane normal direction $\mathbf{n}_\pi$ and plane distance d_π as:

$$\bar{q}_\pi = \begin{bmatrix} \mathbf{q}_v \\ q_4 \end{bmatrix} = \frac{1}{\sqrt{1 + d_\pi^2}} \begin{bmatrix} \mathbf{n}_\pi \\ d_\pi \end{bmatrix} \tag{178}$$

Therefore, we can use the error states $\delta\boldsymbol{\theta}_\pi$ for quaternion to represent the plane, that is:

$$\bar{q}_\pi = \begin{bmatrix} \frac{1}{2}\delta\boldsymbol{\theta}_\pi \\ 1 \end{bmatrix} \otimes \hat{\bar{q}}_\pi \tag{179}$$

Note that the Jacobians w.r.t. the quaternion error states can be written as:

$$\bar{q}_\pi = \hat{\bar{q}}_\pi + \tilde{\bar{q}}_\pi = \begin{bmatrix} \frac{1}{2}\delta\boldsymbol{\theta}_\pi \\ 1 \end{bmatrix} \otimes \hat{\bar{q}}_\pi \tag{180}$$

$$\Rightarrow \tilde{\bar{q}}_\pi = \begin{bmatrix} \frac{1}{2}\delta\boldsymbol{\theta}_\pi \\ 0 \end{bmatrix} \otimes \hat{\bar{q}}_\pi = \begin{bmatrix} \hat{q}_4 \mathbf{I}_3 + \lfloor \hat{\mathbf{q}}_v \rfloor & \hat{\mathbf{q}}_v \\ -\hat{\mathbf{q}}_v^\top & \hat{q}_4 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\delta\boldsymbol{\theta}_\pi \\ 0 \end{bmatrix} \tag{181}$$

$$\Rightarrow \begin{bmatrix} \frac{1}{2}\delta\boldsymbol{\theta}_\pi \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{3\times 1} \\ 1 \end{bmatrix} + \tilde{\bar{q}}_\pi \otimes \hat{\bar{q}}_\pi^{-1} = \begin{bmatrix} \mathbf{0}_{3\times 1} \\ 1 \end{bmatrix} + \begin{bmatrix} \hat{q}_4 \mathbf{I}_3 - \lfloor \hat{\mathbf{q}}_v \rfloor & -\hat{\mathbf{q}}_v \\ \hat{\mathbf{q}}_v^\top & \hat{q}_4 \end{bmatrix} \tilde{\bar{q}}_\pi \tag{182}$$

F.2: Measurement Noise Covariance

Consider a point $\mathbf{p_f}$ on the plane, then we write the plane measurement as the following:

$$\mathbf{z} = \mathbf{h}\left(\bar{q}_\pi, \mathbf{n}_p\right) \tag{183}$$

$$= \mathbf{q}_v^\top \left(\mathbf{p_f} + \mathbf{n}_p\right) - q_4 \tag{184}$$

We linearize the above equation and get:

$$\tilde{\mathbf{z}} \simeq \mathbf{H}_\pi \delta\boldsymbol{\theta}_\pi + \mathbf{H}_n \mathbf{n}_p \tag{185}$$

$$\mathbf{H}_\pi = \frac{\partial \tilde{\mathbf{z}}}{\partial \delta\boldsymbol{\theta}_\pi} = \frac{\partial \tilde{\mathbf{z}}}{\partial \tilde{\bar{q}}_\pi} \frac{\partial \tilde{\bar{q}}_\pi}{\partial \delta\boldsymbol{\theta}_\pi} \tag{186}$$

$$\mathbf{H}_n = \frac{\partial \tilde{\mathbf{z}}}{\partial \mathbf{n}_p} \tag{187}$$

where:

$$\frac{\partial \tilde{\mathbf{z}}}{\partial \tilde{\bar{q}}_\pi} = \begin{bmatrix} \hat{\mathbf{p}}_{\mathbf{f}}^\top & -1 \end{bmatrix} \tag{188}$$

$$\frac{\partial \tilde{\bar{q}}_\pi}{\partial \delta\boldsymbol{\theta}_\pi} = \frac{1}{2} \begin{bmatrix} \hat{q}_4 \mathbf{I}_3 + \lfloor \hat{\mathbf{q}}_v \rfloor \\ -\hat{\mathbf{q}}_v^\top \end{bmatrix} \tag{189}$$

$$\frac{\partial \tilde{\mathbf{z}}}{\partial \mathbf{n}_p} = \hat{\mathbf{q}}_v^\top \tag{190}$$

We can minimize the difference between each point and the quaternion representation to get the optimal plane parameters. After optimization, we can get the measurement covariance, $\mathbf{R}_\pi$, by looping over all measurements to compute the following:

$$\mathbf{R}_\pi = \left(\sum_i \mathbf{H}_{\pi i}^\top \left(\mathbf{H}_{ni} W_i \mathbf{H}_{ni}^\top \right)^{-1} \mathbf{H}_{\pi i} \right)^{-1} \tag{191}$$

F.3: Jacobians for Anchor Factor

Having compressed the point cloud into the quaternion plane representation, we can add it to our factor graph. In order to optimize we need the Jacobians of the measurement in respect to the states that it depends on. We define the following frame of references: $\{L\}$ current LiDAR frame, $\{I\}$ current IMU frame, $\{A\}$ anchored IMU frame and $\{L_a\}$ the anchored LiDAR frame. The measurement function of the anchored plane projected into the current local frame can be summarized as follows:

$$\mathbf{z} = \mathbf{h}\left({}^{La}\bar{q}_\pi, \mathbf{n_R}\right) \tag{192}$$

$$= \mathbf{h}\left({}^I_G\mathbf{R}, {}^G\mathbf{p}_I, {}^A_G\mathbf{R}, {}^G\mathbf{p}_A, {}^{La}\bar{q}_\pi, \mathbf{n_R}\right) \tag{193}$$

where ${}^I_G\mathbf{R}$, ${}^G\mathbf{p}_I$ is the current IMU pose, ${}^A_G\mathbf{R}$, ${}^G\mathbf{p}_A$ is the current anchored IMU pose, ${}^{La}\bar{q}_\pi$ is the plane in the anchor LiDAR frame, and $\mathbf{n_R}$ is noise corrupting the quaternion plane measurement, whose covariance is $\mathbf{R}_\pi$. Therefore, we can have:

$$\mathbf{H}_x = \frac{\partial \tilde{\mathbf{z}}}{\partial \tilde{\mathbf{x}}} = \begin{bmatrix} \frac{\partial \delta^L\boldsymbol{\theta}_\pi}{\partial \delta\boldsymbol{\theta}_I} & \frac{\partial \delta^L\boldsymbol{\theta}_\pi}{\partial {}^G\tilde{\mathbf{p}}_I} & \frac{\partial \delta^L\boldsymbol{\theta}_\pi}{\partial \delta\boldsymbol{\theta}_A} & \frac{\partial \delta^L\boldsymbol{\theta}_\pi}{\partial {}^G\tilde{\mathbf{p}}_A} & \frac{\partial \delta^L\boldsymbol{\theta}_\pi}{\partial {}^{La}\boldsymbol{\theta}_\pi} \end{bmatrix} \tag{194}$$

where:

$$\begin{bmatrix} {}^L\mathbf{n}_\pi \\ {}^Ld_\pi \end{bmatrix} = \begin{bmatrix} {}^L_I\mathbf{R} & \mathbf{0}_{3\times1} \\ -{}^I\mathbf{p}_L^\top & 1 \end{bmatrix} \begin{bmatrix} {}^I_G\mathbf{R} & \mathbf{0}_{3\times1} \\ -{}^G\mathbf{p}_I^\top & 1 \end{bmatrix} \begin{bmatrix} {}^G_A\mathbf{R} & \mathbf{0}_{3\times1} \\ -{}^A\mathbf{p}_G^\top & 1 \end{bmatrix} \begin{bmatrix} {}^I_L\mathbf{R} & \mathbf{0}_{3\times1} \\ -{}^L\mathbf{p}_I^\top & 1 \end{bmatrix} \begin{bmatrix} {}^{La}\mathbf{n}_\pi \\ {}^{La}d_\pi \end{bmatrix} \tag{195}$$

We first compute the Jacobians w.r.t. the current IMU pose ${}^I_G\mathbf{R}$ and ${}^G\mathbf{p}_I$ as:

$$\frac{\partial \delta^L\boldsymbol{\theta}_\pi}{\partial \delta\boldsymbol{\theta}_I} = \frac{\partial \delta^L\boldsymbol{\theta}_\pi}{\partial {}^L\tilde{\bar{q}}_\pi} \frac{\partial {}^L\tilde{\bar{q}}_\pi}{\partial \begin{bmatrix} {}^L\tilde{\mathbf{n}}_\pi \\ {}^L\tilde{d}_\pi \end{bmatrix}} \frac{\partial \begin{bmatrix} {}^L\tilde{\mathbf{n}}_\pi \\ {}^L\tilde{d}_\pi \end{bmatrix}}{\partial \delta\boldsymbol{\theta}_I} \tag{196}$$

$$\frac{\partial \delta^L\boldsymbol{\theta}_\pi}{\partial {}^G\tilde{\mathbf{p}}_I} = \frac{\partial \delta^L\boldsymbol{\theta}_\pi}{\partial {}^L\tilde{\bar{q}}_\pi} \frac{\partial {}^L\tilde{\bar{q}}_\pi}{\partial \begin{bmatrix} {}^L\tilde{\mathbf{n}}_\pi \\ {}^L\tilde{d}_\pi \end{bmatrix}} \frac{\partial \begin{bmatrix} {}^L\tilde{\mathbf{n}}_\pi \\ {}^L\tilde{d}_\pi \end{bmatrix}}{\partial {}^G\tilde{\mathbf{p}}_I} \tag{197}$$

The measurement relationship can be described as:

$$\begin{bmatrix} {}^L\mathbf{n}_\pi \\ {}^Ld_\pi \end{bmatrix} = \begin{bmatrix} {}^L_I\mathbf{R} & \mathbf{0}_{3\times1} \\ -{}^I\mathbf{p}_L^\top & 1 \end{bmatrix} \begin{bmatrix} {}^I_G\mathbf{R} & \mathbf{0}_{3\times1} \\ -{}^G\mathbf{p}_I^\top & 1 \end{bmatrix} \begin{bmatrix} {}^G\mathbf{n}_\pi \\ {}^Gd_\pi \end{bmatrix} \tag{198}$$

where:

$$\frac{\partial \delta^L\boldsymbol{\theta}_\pi}{\partial {}^L\tilde{\bar{q}}_\pi} = 2 \begin{bmatrix} {}^L\hat{q}_4\mathbf{I}_3 - \lfloor {}^L\hat{\mathbf{q}}_v \rfloor & -{}^L\hat{\mathbf{q}}_v \end{bmatrix} \tag{199}$$

$$\frac{\partial {}^L\tilde{\bar{q}}_\pi}{\partial \begin{bmatrix} {}^L\tilde{\mathbf{n}}_\pi \\ {}^L\tilde{d}_\pi \end{bmatrix}} = \frac{1}{[1 + {}^Ld_\pi^2]^{\frac{3}{2}}} \begin{bmatrix} \left(1 + {}^Ld_\pi^2\right)\mathbf{I}_3 & -{}^Ld_\pi{}^L\mathbf{n}_\pi \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix} \tag{200}$$

$$\frac{\partial \begin{bmatrix} {}^L\tilde{\mathbf{n}}_\pi \\ {}^L\tilde{d}_\pi \end{bmatrix}}{\partial \delta\boldsymbol{\theta}_I} = \begin{bmatrix} {}^L_I\mathbf{R} & \mathbf{0}_{3\times1} \\ -{}^I\mathbf{p}_L^\top & 1 \end{bmatrix} \begin{bmatrix} \lfloor {}^I_G\mathbf{R}{}^G\mathbf{n}_\pi \rfloor \\ \mathbf{0}_{1\times3} \end{bmatrix} \tag{201}$$

$$\frac{\partial \begin{bmatrix} {}^L\tilde{\mathbf{n}}_\pi \\ {}^L\tilde{d}_\pi \end{bmatrix}}{\partial {}^G\tilde{\mathbf{p}}_I} = \begin{bmatrix} {}^L_I\mathbf{R} & \mathbf{0}_{3\times1} \\ -{}^I\mathbf{p}_L^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{0}_3 \\ -{}^G\mathbf{n}_\pi^\top \end{bmatrix} \tag{202}$$

We then compute the Jacobians w.r.t. the current IMU pose $_G^A\mathbf{R}$ and $^G\mathbf{p}_A$ as:

$$\frac{\partial \delta^L\boldsymbol{\theta}_\pi}{\partial \delta\boldsymbol{\theta}_A} = \frac{\partial \delta^L\boldsymbol{\theta}_\pi}{\partial ^L\tilde{\bar{q}}_\pi} \frac{\partial ^L\tilde{\bar{q}}_\pi}{\partial \begin{bmatrix} ^L\tilde{\mathbf{n}}_\pi \\ ^L\tilde{d}_\pi \end{bmatrix}} \frac{\partial \begin{bmatrix} ^L\tilde{\mathbf{n}}_\pi \\ ^L\tilde{d}_\pi \end{bmatrix}}{\partial \delta\boldsymbol{\theta}_A} \tag{203}$$

$$\frac{\partial \delta^L\boldsymbol{\theta}_\pi}{\partial ^G\tilde{\mathbf{p}}_A} = \frac{\partial \delta^L\boldsymbol{\theta}_\pi}{\partial ^L\tilde{\bar{q}}_\pi} \frac{\partial ^L\tilde{\bar{q}}_\pi}{\partial \begin{bmatrix} ^L\tilde{\mathbf{n}}_\pi \\ ^L\tilde{d}_\pi \end{bmatrix}} \frac{\partial \begin{bmatrix} ^L\tilde{\mathbf{n}}_\pi \\ ^L\tilde{d}_\pi \end{bmatrix}}{\partial ^G\tilde{\mathbf{p}}_A} \tag{204}$$

The measurement relationship can be described as:

$$\begin{bmatrix} ^L\mathbf{n}_\pi \\ ^Ld_\pi \end{bmatrix} = \begin{bmatrix} _I^L\mathbf{R} & \mathbf{0}_{3\times1} \\ -^I\mathbf{p}_L^\top & 1 \end{bmatrix} \begin{bmatrix} _G^I\mathbf{R} & \mathbf{0}_{3\times1} \\ -^G\mathbf{p}_I^\top & 1 \end{bmatrix} \begin{bmatrix} _A^G\mathbf{R} & \mathbf{0}_{3\times1} \\ -^A\mathbf{p}_G^\top & 1 \end{bmatrix} \begin{bmatrix} ^A\mathbf{n}_\pi \\ ^Ad_\pi \end{bmatrix} \tag{205}$$

where:

$$\frac{\partial \delta^L\boldsymbol{\theta}_\pi}{\partial ^L\tilde{\bar{q}}_\pi} = 2 \begin{bmatrix} ^L\hat{q}_4\mathbf{I}_3 - \lfloor ^L\hat{\mathbf{q}}_v \rfloor & -^L\hat{\mathbf{q}}_v \end{bmatrix} \tag{206}$$

$$\frac{\partial ^L\tilde{\bar{q}}_\pi}{\partial \begin{bmatrix} ^L\tilde{\mathbf{n}}_\pi \\ ^L\tilde{d}_\pi \end{bmatrix}} = \frac{1}{[1 + ^Ld_\pi^2]^{\frac{3}{2}}} \begin{bmatrix} \left(1 + ^Ld_\pi^2\right)\mathbf{I}_3 & -^Ld_\pi{}^L\mathbf{n}_\pi \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix} \tag{207}$$

$$\frac{\partial \begin{bmatrix} ^L\tilde{\mathbf{n}}_\pi \\ ^L\tilde{d}_\pi \end{bmatrix}}{\partial \delta\boldsymbol{\theta}_A} = \begin{bmatrix} _I^L\mathbf{R} & \mathbf{0}_{3\times1} \\ -^I\mathbf{p}_L^\top & 1 \end{bmatrix} \begin{bmatrix} _G^I\mathbf{R} & \mathbf{0}_{3\times1} \\ -^G\mathbf{p}_I^\top & 1 \end{bmatrix} \begin{bmatrix} -_A^G\mathbf{R}\lfloor ^A\mathbf{n}_\pi \rfloor \\ ^A\mathbf{n}_\pi^\top\lfloor _G^A\mathbf{R}^G\mathbf{p}_A \rfloor \end{bmatrix} \tag{208}$$

$$\frac{\partial \begin{bmatrix} ^L\tilde{\mathbf{n}}_\pi \\ ^L\tilde{d}_\pi \end{bmatrix}}{\partial ^G\tilde{\mathbf{p}}_A} = \begin{bmatrix} _I^L\mathbf{R} & \mathbf{0}_{3\times1} \\ -^I\mathbf{p}_L^\top & 1 \end{bmatrix} \begin{bmatrix} _G^I\mathbf{R} & \mathbf{0}_{3\times1} \\ -^G\mathbf{p}_I^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{0}_3 \\ ^A\mathbf{n}_\pi^\top{}_G^A\mathbf{R} \end{bmatrix} \tag{209}$$

Finally, we compute the Jacobians w.r.t. the plane state in the anchored LiDAR frame $^{La}\bar{q}_\pi$ as:

$$\frac{\partial \delta^L\boldsymbol{\theta}_\pi}{\partial \delta^G\boldsymbol{\theta}_\pi} = \frac{\partial \delta^L\boldsymbol{\theta}_\pi}{\partial ^L\tilde{\bar{q}}_\pi} \frac{\partial ^C\tilde{\bar{q}}_\pi}{\partial \begin{bmatrix} ^L\tilde{\mathbf{n}}_\pi \\ ^L\tilde{d}_\pi \end{bmatrix}} \frac{\partial \begin{bmatrix} ^L\tilde{\mathbf{n}}_\pi \\ ^L\tilde{d}_\pi \end{bmatrix}}{\partial \begin{bmatrix} ^{La}\tilde{\mathbf{n}}_\pi \\ ^{La}\tilde{d}_\pi \end{bmatrix}} \frac{\partial \begin{bmatrix} ^{La}\tilde{\mathbf{n}}_\pi \\ ^{La}\tilde{d}_\pi \end{bmatrix}}{\partial ^{La}\tilde{\bar{q}}_\pi} \frac{\partial ^{La}\tilde{\bar{q}}_\pi}{\partial \delta^{La}\boldsymbol{\theta}_\pi} \tag{210}$$

The measurement relationship can be described as:

$$\begin{bmatrix} ^L\mathbf{n}_\pi \\ ^Ld_\pi \end{bmatrix} = \begin{bmatrix} _I^L\mathbf{R} & \mathbf{0}_{3\times1} \\ -^I\mathbf{p}_L^\top & 1 \end{bmatrix} \begin{bmatrix} _G^I\mathbf{R} & \mathbf{0}_{3\times1} \\ -^G\mathbf{p}_I^\top & 1 \end{bmatrix} \begin{bmatrix} _A^G\mathbf{R} & \mathbf{0}_{3\times1} \\ -^A\mathbf{p}_G^\top & 1 \end{bmatrix} \begin{bmatrix} _L^I\mathbf{R} & \mathbf{0}_{3\times1} \\ -^L\mathbf{p}_I^\top & 1 \end{bmatrix} \begin{bmatrix} ^{La}\mathbf{n}_\pi \\ ^{La}d_\pi \end{bmatrix} \tag{211}$$

where:

$$\frac{\partial \delta^L \boldsymbol{\theta}_\pi}{\partial^L \tilde{\hat{q}}_\pi} = 2 \begin{bmatrix} {}^L \hat{q}_4 \mathbf{I}_3 - \lfloor {}^L \hat{\mathbf{q}}_v \rfloor & -{}^L \hat{\mathbf{q}}_v \end{bmatrix} \tag{212}$$

$$\frac{\partial^L \tilde{\hat{q}}_\pi}{\partial \begin{bmatrix} {}^L \tilde{\mathbf{n}}_\pi \\ {}^L \tilde{d}_\pi \end{bmatrix}} = \frac{1}{[1 + {}^L d_\pi^2]^{\frac{3}{2}}} \begin{bmatrix} \left(1 + {}^L d_\pi^2\right) \mathbf{I}_3 & -{}^L d_\pi {}^L \mathbf{n}_\pi \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \tag{213}$$

$$\frac{\partial \begin{bmatrix} {}^L \tilde{\mathbf{n}}_\pi \\ {}^L \tilde{d}_\pi \end{bmatrix}}{\partial \begin{bmatrix} {}^{La} \tilde{\mathbf{n}}_\pi \\ {}^{La} \tilde{d}_\pi \end{bmatrix}} = \begin{bmatrix} {}^L_I \mathbf{R} & \mathbf{0}_{3\times 1} \\ -{}^I \mathbf{p}_L^\top & 1 \end{bmatrix} \begin{bmatrix} {}^I_G \mathbf{R} & \mathbf{0}_{3\times 1} \\ -{}^G \mathbf{p}_I^\top & 1 \end{bmatrix} \begin{bmatrix} {}^G_A \mathbf{R} & \mathbf{0}_{3\times 1} \\ -{}^A \mathbf{p}_G^\top & 1 \end{bmatrix} \begin{bmatrix} {}^I_L \mathbf{R} & \mathbf{0}_{3\times 1} \\ -{}^L \mathbf{p}_I^\top & 1 \end{bmatrix} \tag{214}$$

$$\frac{\partial \begin{bmatrix} {}^{La} \tilde{\mathbf{n}}_\pi \\ {}^{La} \tilde{d}_\pi \end{bmatrix}}{\partial^{La} \tilde{\hat{q}}_\pi} = \frac{1}{[{}^{La} \mathbf{q}_v^\top {}^{La} \mathbf{q}_v]^{\frac{3}{2}}} \begin{bmatrix} -\lfloor {}^{La} \mathbf{q}_v \rfloor^2 & \mathbf{0}_{3\times 1} \\ -{}^{La} q_4 {}^{La} \mathbf{q}_v^\top & {}^{La} \mathbf{q}_v^\top {}^{La} \mathbf{q}_v \end{bmatrix} \tag{215}$$

$$\frac{\partial^{La} \tilde{\hat{q}}_\pi}{\partial \delta^{La} \boldsymbol{\theta}_\pi} = \frac{1}{2} \begin{bmatrix} {}^{La} \hat{q}_4 \mathbf{I}_3 + \lfloor {}^{La} \hat{\mathbf{q}}_v \rfloor \\ -{}^{La} \hat{\mathbf{q}}_v^\top \end{bmatrix} \tag{216}$$

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